A RELATIVISTIC QUARK MODEL FOR MESONS BASED ON NUMERICAL SOLUTIONS OF THE BETHE-SALPETER EQUATION

by

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ABSTRACT

A RELATIVISTIC QUARK MODEL FOR MESONS

BASED ON NUMERICAL SOLUTIONS OF THE BETHE-SALPETER EQUATIONS

by Alan H. Guth

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Numerical solutions to the Bethe-Salpeter equation are obtained for pseudoscalar and vector bound states of a deeply bound equal mass spin-$\frac{1}{2}$ quark-antiquark pair, with either a scalar, pseudoscalar, or neutral vector exchange interaction. The interaction function corresponds to single particle exchange, with the addition of either one or two regulating terms. It is found that the second regulator allows the internal quark momentum to be much smaller than the quark mass, but that the spinor structure of the wavefunction remains highly relativistic.

Only the scalar interaction can account for the observed spectrum of states. The pseudoscalar interaction yields a vector state of lower mass than the pseudoscalar state, and the vector interaction leads to a vector state which lies approximately one quark mass above the pseudoscalar state. The λ quark is taken as slightly heavier than the p and n, and the perturbation treatment of the mass difference leads to a quadratic mass formula. The $J^{PC} = 0^{+-}$ daughters of the vector mesons are also studied, and it is found that they are massive ($\sim$ 2 Bev) ghost states.

The decay amplitudes for π, $K \to \mu\nu$ are calculated, and it is found, independent of parameters, that $f_\pi \approx f_K$ for either a scalar or vector interaction, in agreement with experiment. The amplitudes for ρ^0, ω, $\phi \to e^+e^-$, $\mu^+\mu^-$ are also calculated, but in this case the ratios (again parameter independent) are in minor discrepancy with experiment. The magnetic moments of the vector mesons and the amplitudes for magnetic transitions such as $\omega \to \pi^0\gamma$ are calculated, but with significant restrictions. The magnetic moments of the vector mesons have the same (trivial) ratios to each other as in the nonrelativistic model, but they are much larger than the sum of the quark magnetic moments. The amplitude for magnetic transitions, however, is related to the quark magnetic moments in very nearly the nonrelativistic ratio.

The model is also used to obtain parameter dependent predictions about masses and decay amplitudes. These predictions are not experimentally correct, but are within an order of magnitude.

Thesis Supervisor: Francis E. Low

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TABLE OF CONTENTS

INTRODUCTION

The quark hypothesis, proposed by Gell-Mann[1] and Zweig [2], has led to a great deal of success in understanding the properties of hadrons[24]. Most of these successes rely on nonrelativistic calculations, using simple assumptions such as the additivity of quark amplitudes, and simple approximations such as the setting of wavefunction overlap integrals equal to unity. Inasmuch as quarks have not yet been found, it is unclear why these non-relativistic calculations are so successful. One possibility is that quarks are not physical particles at all, but are rather mathematical entities which occur in some as yet unknown theory which gives approximately nonrelativistic results. The other possibility is that quarks are very massive, but that somehow the deeply bound states have nonrelativistic properties. This thesis will explore the latter possibility. We will try to reproduce the success of the nonrelativistic quark model using the fully relativistic formalism of the Bethe-Salpeter equation to describe deeply bound states. Our attention will be confined to the low-lying pseudoscalar and vector mesons.

Morpurgo has suggested[3] that the motion of the quarks need not be relativistic, provided that they are bound by a long range force. The quarks can then move nonrelativistically at the bottom of a deep potential well. In the Bethe-Salpeter equation, the interaction is expressed by a function which represents, in perturbation theory, the sum of all irreducible Feynman diagrams. Since we cannot carry out this sum, we will use instead a phenomenological interaction function which incorporates Morpurgo's suggestion. In particular, we will modify the one particle exchange propagator by the addition of one regulating term:

$$\frac{1}{q^2+\mu^2} \longrightarrow \frac{1}{q^2+\mu^2} - \frac{1}{q^2+\Lambda^2} \quad .$$

Λ is not a cutoff mass to be assigned a large value, but is rather a range parameter which is intended to have a small value. The above interaction is found to lead to bound states with average internal quark momenta comparable to the quark mass, so we modify the interaction further by adding a second regulator:

$$\frac{1}{q^2+\mu^2} \longrightarrow \frac{1}{q^2+\mu^2} - \frac{1}{q^2+\Lambda^2} - \frac{\Lambda^2-\mu^2}{(q^2+\Lambda^2)^2} \quad .$$

With this interaction one can find deeply bound states with small quark momenta, but the spinor structure of the deeply bound wavefunction remains highly relativistic.

We will use interactions which are scalar, pseudoscalar, or neutral vector. Since the interaction represents the sum of irreducible diagrams, this choice of spin and parity does not necessarily correspond to the spin and parity of the gluon which appears in the Lagrangian. Any realistic interaction function will contain terms of these forms and others, but we will just treat these terms one at a time.

The thesis is divided into three parts. The first part reviews the development of the Bethe-Salpeter formalism. The Bethe-Salpeter equation is derived, and it is shown how to normalize the wavefunctions and use them to calculate amplitudes and perturbation expressions. In Part II, it is shown how the Bethe-Salpeter equation for deeply bound states can be reduced and solved numerically. The results are presented in Part III.

PART I:

REVIEW OF BETHE-SALPETER FORMALISM

A. INTRODUCTION

This part will comprise a review of the Bethe-Salpeter formalism for treating bound state problems in a fully relativistic, fully quantized field theory. The results and the notation which will be used in the following parts are established here. Nothing in this part is really new, so readers familiar with the subject may wish to skip it.

The basis of the Bethe-Salpeter formalism, including the Bethe-Salpeter equations for the four-point Green's function and for the bound state wavefunction, was first established by Salpeter and Bethe [4] and Gell-Mann and Low [5]. The problem of normalizing the wavefunctions and calculating matrix elements involving bound states was solved by Mandelstam [6]. The normalization technique used here was derived by Lurié, MacFarlane, and Takahashi [7], and later emphasized by Llewellyn Smith [8].

B. DEFINITION OF THE BETHE-SALPETER WAVEFUNCTION.

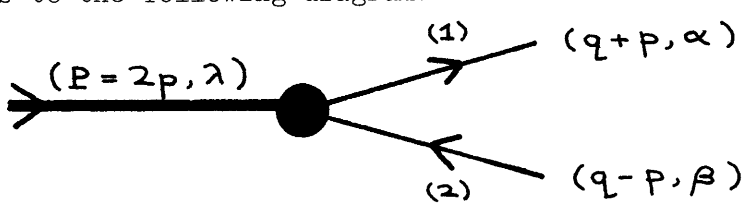
We define the Bethe-Salpeter (B-S) wavefunction for the bound state $|P,\lambda\rangle$ of a fermion-antifermion (quark-antiquark) pair as

$$\chi_{\alpha\beta}(p,q,\lambda) \equiv (2\pi)^{3/2}\int d^4x\, e^{-iq\cdot x} \times \langle 0|T\{\psi_\alpha^{(1)}(\tfrac{x}{2})\,\overline{\psi}_\beta^{(2)}(-\tfrac{x}{2})\}|P,\lambda\rangle, \tag{I-B-1}$$

where $|0\rangle$ is the physical vacuum, $|P,\lambda\rangle$ is the bound state (with spin index λ and four-momentum $P_\mu \equiv 2p_\mu$), T is the Wick time-ordered product, and the ψ's are fields in the Heisenberg picture (which may or may not be associated with the same particle). Throughout this thesis, state vectors will be normalized covariantly:

$$\langle P',\lambda'|P,\lambda\rangle = 2P_0\,\delta^3(\vec{P}'-\vec{P})\,\delta_{\lambda'\lambda}. \tag{I-B-2}$$

The two Dirac indices of χ will sometimes be suppressed. $\chi_{\alpha\beta}(p,q,\lambda)$ corresponds to the following diagram:

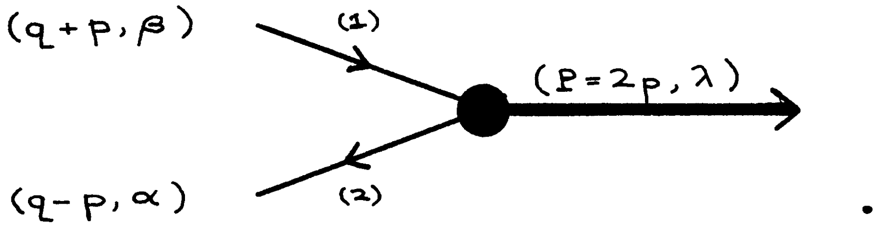


.

The conjugate B-S wavefunction is defined by

$$\overline{\chi}_{\alpha\beta}(p,q,\lambda) \equiv (2\pi)^{3/2}\int d^4x\, e^{iq\cdot x} \times \langle P,\lambda|T\{\overline{\psi}_\beta^{(1)}(\tfrac{x}{2})\,\psi_\alpha^{(2)}(-\tfrac{x}{2})\}|0\rangle, \tag{I-B-3}$$

and corresponds to the diagram


(q+p, β)
(1)
(P=2p, λ)
(q−p, α)
(2)

C. SPECTRAL REPRESENTATION OF THE BETHE-SALPETER WAVEFUNCTION

The spectral representation is useful to exhibit the singularities of the B-S wavefunctions, and to derive the relationship between $\chi(p,q,\lambda)$ and $\bar{\chi}(p,q,\lambda)$.

By inserting intermediate states in the usual way, assuming translation invariance and the completeness of physical states, one can show that

$$\chi(p,q,\lambda) = i\sqrt{2\pi}\int dK_0 \left\{ \frac{\rho_1(\vec{q}+\vec{p}, K_0; \lambda)}{q_0 - K_0 + p_0 + i\epsilon} + \frac{\rho_2(\vec{p}-\vec{q}, K_0; \lambda)}{q_0 + K_0 - p_0 - i\epsilon} \right\}, \tag{I-C-1}$$

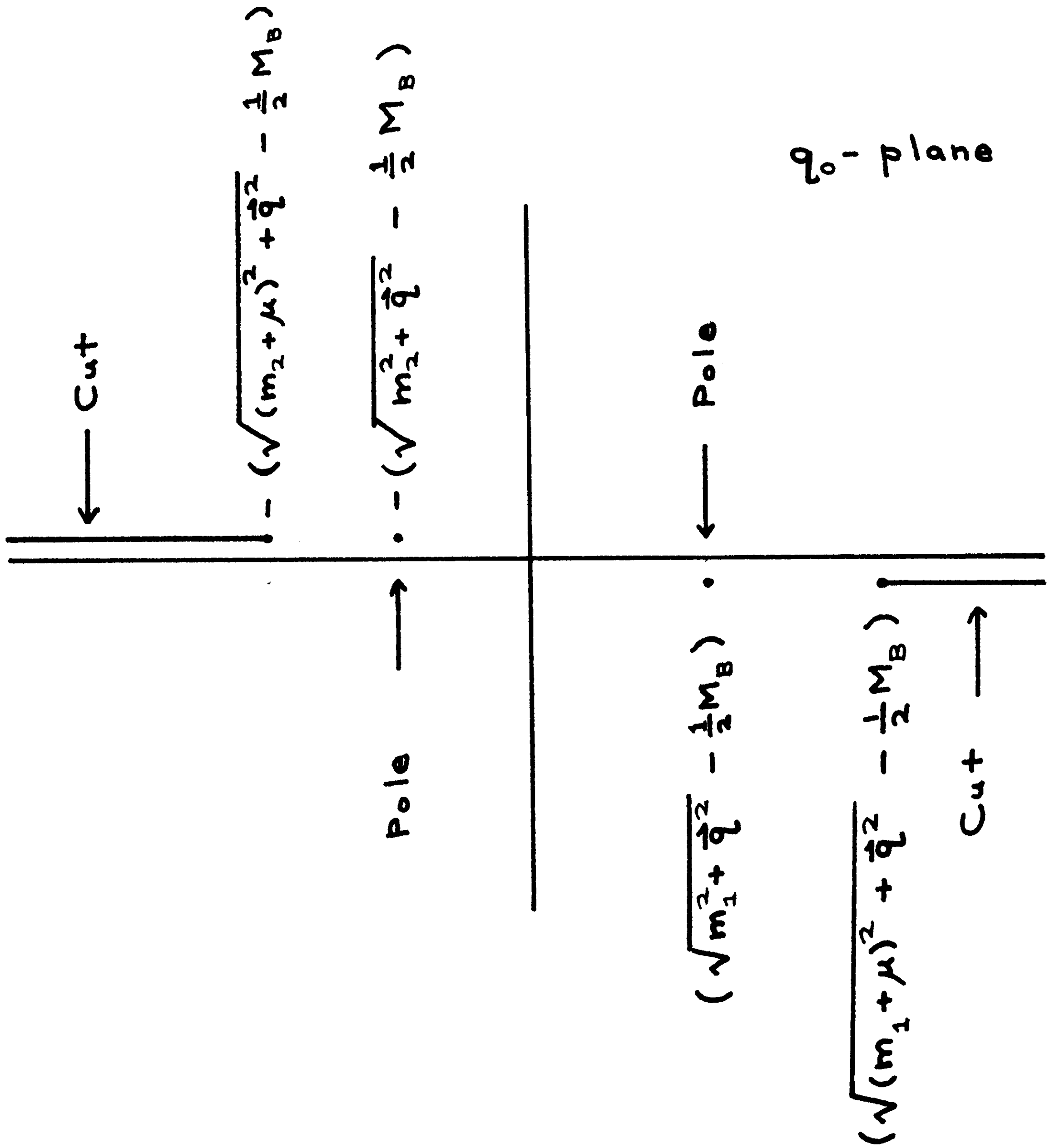
where

$$\rho_{1\alpha\beta}(K;\lambda) = (2\pi)^4 \sum_n \delta^4(p_n - K) \langle 0 | \psi_\alpha^{(1)}(0) | n \rangle \times \langle n | \bar{\psi}_\beta^{(2)}(0) | P, \lambda \rangle,$$

and

$$\rho_{2\alpha\beta}(K;\lambda) = (2\pi)^4 \sum_n \delta^4(p_n - K) \langle 0 | \bar{\psi}_\beta^{(2)}(0) | n \rangle \times \langle n | \psi_\alpha^{(1)}(0) | P, \lambda \rangle. \tag{I-C-2}$$

In the rest frame of the bound state, where $p = (0, \frac{i}{2} M_B)$, the singularities in the q_0-plane are as shown below:


q_0- plane
Cut
$-(\sqrt{(m_2+\mu)^2+\vec{q}^2} - \frac{1}{2}M_B)$
$-(\sqrt{m_2^2+\vec{q}^2} - \frac{1}{2}M_B)$
Pole
Pole
$(\sqrt{m_1^2+\vec{q}^2} - \frac{1}{2}M_B)$
$(\sqrt{(m_1+\mu)^2+\vec{q}^2} - \frac{1}{2}M_B)$
Cut

m_1 and m_2 are the masses of the particles associated with the fields $\psi^{(1)}(x)$ and $\psi^{(2)}(x)$, respectively. (More precisely, m_i is the mass of the lightest particle for which the matrix element $\langle 0 | \psi^{(i)}(x) | m_i \rangle$ is non-vanishing.) μ is the mass of the lightest neutral particle. (More precisely, $\mu + m_i$ is the mass of the lightest two particle state for which the matrix element $\langle 0 | \psi^{(i)}(x) | \mu m_i \rangle$ is non-vanishing.)

The conjugate wavefunction can similarly be espressed as

$$\bar{\chi}(p,q,\lambda) = -i\sqrt{2\pi}\int dK \left\{ \frac{\bar{\rho}_1(\vec{p}+\vec{q}, K_0; \lambda)}{q_0 - K_0 + p_0 + i\epsilon} \right.$$

$$\left. \frac{\bar{\rho}_2(\vec{p}-\vec{q}, K_0; \lambda)}{q_0 - K_0 - p_0 - i\epsilon} \right\}, \tag{I-C-3}$$

where

$$\bar{\rho}_i(K;\lambda) = \gamma_4 \rho_i^{+}(K,\lambda)\gamma_4 . \tag{I-C-4}$$

(The $+$ is defined by

$$A^{+}_{\alpha\beta} \equiv A^{*}_{\beta\alpha},$$

where α, β are Dirac indices, and the * indicates complex conjugation.) Thus the poles and cuts of $\bar{\chi}(p,q,\lambda)$ coincide exactly with those of $\chi(p,q,\lambda)$. For complex q_0, the two functions are related by

$$\bar{\chi}(p,\vec{q},q_0,\lambda) = \gamma_4 \chi^{+}(p,\vec{q},q_0^{*},\lambda)\gamma_4 . \tag{I-C-5}$$

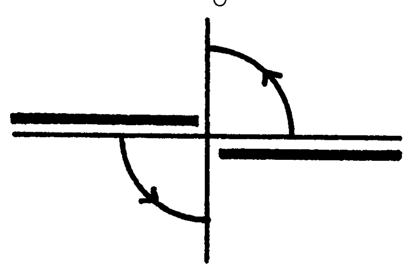
For real q_0, both functions are to be evaluated by approaching the positive q_0-axis from above and the negative q_0-axis from below.

D. THE WICK ROTATION

A great deal of mathematical simplification results from an analytic continuation which was first introduced by G.C. Wick [9]. $\chi(p,q,\lambda)$ is an analytic function of q_o, with singularities whose locations were shown in the previous section. Provided that

$$\begin{aligned} M_B &< 2m_1 \\ M_B &< 2m_2, \end{aligned} \tag{I-D-1}$$

it can be seen that all of the singularities are located either just below the positive real axis or just above the negative real axis. (By redefining q_o by a translation, this statement can be made true for any bound state.) So without crossing any singularities, it is possible to analytically continue the function to the imaginary q_o-axis.



Then $q_4 = iq_o$ will be real, and the vector (q_1, q_2, q_3, q_4) will be a real Euclidean vector. Such Euclidean vectors will be denoted in this thesis by a bar above the symbol -- i.e., $\bar{q}$.

In the rest frame of the bound state, where $p = (0, \frac{i}{2} M_B)$, it is then convenient to define

$$\bar{p} = -ip = (\vec{0}, \tfrac{1}{2} M_B). \tag{I-D-2}$$

Then $\bar{p}$ is also a real 4-vector, provided one stays in the rest frame.

The Wick rotation is very useful in simplifying the integration of expressions involving the B-S wavefunction. In such cases one must verify that other factors in the integrand do not introduce singularities which interfere, and that the contour at infinity can be neglected. If the asymptotic behavior of the integrand is given by a power of q_0, then the condition that the contour at infinity can be neglected is the same as the condition that the integral converge. For the wavefunctions which will be used in this thesis, the contours at infinity will give no contribution.

Note that one picks up a factor of i in the change of integration variables.

$$\int d^4q = \int d^3q \int_{-\infty}^{\infty} dq_0 \tag{I-D-3}$$

$$\xrightarrow[\text{continuation}]{\text{analytic}} \int d^3q \int_{-i\infty}^{i\infty} dq_0$$

$$= -i \int d^3q \int_{\infty}^{-\infty} dq_4 = i \int d^4\bar{q} \, .$$

E. TRANSFORMATION PROPERTIES OF THE BETHE-SALPETER WAVEFUNCTION

1. PROPER LORENTZ TRANSFORMATIONS:

Under proper Lorentz transformations Λ, the fields transform as

$$U^{+}(\Lambda)\psi(x)U(\Lambda) = \mathcal{D}(\Lambda)\psi(\Lambda^{-1}x), \qquad \text{(I-E-1)}$$

where the $\mathcal{D}(\Lambda)$ are representation matrices of the Lorentz group which have the properties

$$\mathcal{D}^{-1}(\Lambda) = \gamma_4 \mathcal{D}^{+}(\Lambda)\gamma_4 \qquad \text{(I-E-2)}$$

and

$$\mathcal{D}^{-1}(\Lambda)\gamma_\mu \mathcal{D}(\Lambda) = \Lambda_{\mu\nu}\gamma_\nu . \qquad \text{(I-E-3)}$$

The transformation of single particle states is rather complicated in the usual canonical or helicity formalisms. Since the bound states have integral spin, it is possible to use polarization tensors which have much simpler transformation properties. A state of spin j is denoted by j tensor indices:

$$|P;\mu_1,\mu_2,\cdots,\mu_j\rangle .$$

These states are not linearly independent, but rather are traceless and symmetric in the indices, and they have the property that

$$P_{\mu_1}|P;\mu_1,\mu_2,\ldots,\mu_j\rangle = 0 . \qquad \text{(I-E-4)}$$

Under Lorentz transformations,

$$U(\Lambda)|P;\mu_1,\mu_2,\ldots,\mu_j\rangle = |\Lambda P;\nu_1,\nu_2,\ldots,\nu_j\rangle \Lambda_{\nu_1\mu_1}\Lambda_{\nu_2\mu_2}\cdots\Lambda_{\nu_j\mu_j} \tag{I-E-5}$$

Superpositions of these basis states may be represented compactly by the symmetric polarization tensor $e_{\mu_1\mu_2\cdots\mu_j}$:

$$|P;e\rangle = |P;\mu_1,\mu_2,\ldots,\mu_j\rangle e_{\mu_1\mu_2\cdots\mu_j}, \tag{I-E-6}$$

where

$$P_{\mu_1} e_{\mu_1\mu_2\cdots\mu_j} = 0 .$$

Under Lorentz transformations,

$$U(\Lambda)|P;e\rangle = |\Lambda P;\Lambda e\rangle, \tag{I-E-7}$$

where we use the shorthand Λe for

$$(\Lambda e)_{\mu_1\mu_2\cdots\mu_j} = \Lambda_{\mu_1\nu_1}\Lambda_{\mu_2\nu_2}\cdots\Lambda_{\mu_j\nu_j} e_{\nu_1\nu_2\cdots\nu_j} . \tag{I-E-8}$$

The Lorentz transformation properties of the B-S wavefunction can then be shown to be

$$\chi(p,q,e) = \mathcal{D}^{-1}(\Lambda)\,\chi(\Lambda p,\Lambda q,\Lambda e)\,\mathcal{D}(\Lambda). \tag{I-E-9}$$

2. PARITY:

If parity is a good symmetry, then it provides a condition on the B-S wavefunction. Under a parity transformation I_P, a Dirac field transforms as

$$I_P^{+} \psi^{(i)}(\vec{x}, t) I_P = \eta_P^{(i)} \gamma_4 \psi^{(i)}(-\vec{x}, t), \qquad \text{(I-E-10)}$$

where $\eta_P^{(i)}$ is the parity of the particle associated with the field, which we take as ±1. If the bound state has parity $\eta_P^{(B)}$, then it transforms as

$$I_P |\vec{P}, P_0; e\rangle = \eta_P^{(B)} (-1)^j |-\vec{P}, P_0; e^{(P)}\rangle, \qquad \text{(I-E-11)}$$

where

$$e^{(P)}_{\mu_1 \mu_2 \cdots \mu_j} = (-1)^S e_{\mu_1 \mu_2 \cdots \mu_j} . \qquad \text{(I-E-12)}$$

where S is the number of space indices. By applying these relations to the definition of the B-S wavefunction, it follows that

$$\begin{aligned} \chi(p, q, e) = {} & \eta_P^{(B)} \eta_P^{(1)} \eta_P^{(2)} (-1)^j \\ & \times \gamma_4 \chi(-\vec{p}, p_0; -\vec{q}, q_0; e^{(P)}) \gamma_4 . \end{aligned} \qquad \text{(I-E-13)}$$

3. CHARGE CONJUGATION:

If the charge conjugation is a good symmetry, then it provides a condition on the B-S wavefunction for a bound state of a fermion and its own antiparticle. Under a charge conjugation transformation I_C, a Dirac field transforms as

$$I_C^{+}\psi_\alpha^{(i)}(x)\, I_C = \eta_C C_{\alpha\beta}\bar{\psi}_\beta^{(i)}(x), \qquad \text{(I-E-14)}$$

where η_C is a phase factor and

$$C = i\gamma_4\gamma_2 . \qquad \text{(I-E-15)}$$

If the bound state has charge conjugation number $\eta_C^{(B)} = \pm 1$, then

$$I_C\, |P; e\rangle = \eta_C^{(B)} |P; e\rangle . \qquad \text{(I-E-16)}$$

It follows that

$$\chi(p, q, e) = \eta_C^{(B)} C\, \chi^T(p, -q, e)\, C^{+} , \qquad \text{(I-E-17)}$$

where the superscript T denotes the transposition of the Dirac indices.

4. TIME REVERSAL:

If time reversal invariance holds, it provides a relationship between the B-S wavefunction and its complex conjugate. Under the (anti-unitary) time reversal operator I_T, the Dirac field transforms as

$$I_T^{-1}\psi^{(i)}(\vec{x}, t)\, I_T = \eta_T^{(i)} T\, \psi^{(i)}(\vec{x}, -t), \qquad \text{(I-E-18)}$$

where $\eta_T^{(i)}$ is a phase factor and

$$T = i\gamma_1\gamma_2 . \qquad \text{(I-E-19)}$$

The bound state transforms as

$$I_T |\vec{P}, P_0; e\rangle = \eta_T^{(B)} |-\vec{P}, P_0; e^*\rangle. \qquad \text{(I-E-20)}$$

Applying these relations to the B-S wavefunction, one obtains

$$\chi(p, q, e) = -\eta_T^{(B)*}\eta_T^{(1)*}\eta_T^{(2)} \times T^{-1}\gamma_4 \bar{\chi}^T(-\vec{p}, p_0; -\vec{q}, q_0; e^*)\gamma_4 T. \qquad \text{(I-E-21)}$$

Using relation (I-C-5) for $\bar{\chi}$, this becomes

$$\chi(p, q, e) = -\eta_T^{(B)*}\eta_T^{(1)*}\eta_T^{(2)} \times T^{-1}\chi^*(-\vec{p}, p_0; -\vec{q}, q_0^*; e^*)T. \qquad \text{(I-E-22)}$$

To exploit this relation on the real axis, it is useful to invoke parity invariance as expressed by eq.(I-E-13) to reverse $\vec{p}$ and $\vec{q}$.

$$\chi(p, q, e) = -\eta_T^{(B)*}\eta_T^{(1)*}\eta_T^{(2)}\eta_P^{(B)}\eta_P^{(1)}\eta_P^{(2)}(-1)^j \times T^{-1}\gamma_4\chi^*(p; \vec{q}, q_0^*; e^{(P)*})\gamma_4 T. \qquad \text{(I-E-23)}$$

For q_0 on the real axis between the two cuts, q_0^* on the right hand side may be written as q_0. Note that the over-all phase is arbitrary, as $\eta_T^{(B)}$ may be adjusted by redefining the phase of the ket $|P, e\rangle$.

Another important region of the complex plane is the Wick-rotated region, defined by $q_4 = iq_0$ real. Consider the B-S wavefunction in the rest frame ($\vec{p} = 0$) of the bound state, and consider a neutral bound state for which charge conjugation invariance (eq. (I-E-17)) applies. Then

$$\chi(p, q, e) = -\eta_T^{(B)*}\eta_C^{(B)}\gamma_5\chi^+(p, q, e^*)\gamma_5. \qquad \text{(I-E-24)}$$

F. EXPANSION OF THE BETHE-SALPETER WAVEFUNCTION IN LORENTZ INVARIANT FUNCTIONS

Since the B-S wavefunction is a 4 x 4 matrix, it can be expanded in the 16 basis matrices 1, γ_5, γ_μ, $\gamma_\mu\gamma_5$, and $\sigma_{\mu\nu}$:

$$\chi(p,q,e) = \chi^{(S)}(p,q,e) + \chi^{(P)}(p,q,e)\gamma_5 + \chi^{(V)}_\mu(p,q,e)\gamma_\mu + \chi^{(A)}_\mu(p,q,e)\gamma_\mu\gamma_5 + \chi^{(T)}_{\mu\nu}(p,q,e)\sigma_{\mu\nu} . \tag{I-F-1}$$

Under Lorentz transformations, one can apply eqs. (I-E-9) and (I-E-3) to show that each tensor function above transforms as expected:

$$\begin{aligned} \chi^{(S)}_{(P)}(\Lambda p,\Lambda q,\Lambda e) &= \chi^{(S)}_{(P)}(p,q,e) \\ \chi^{(V)}_{(A)\mu}(\Lambda p,\Lambda q,\Lambda e) &= \Lambda_{\mu\nu}\chi^{(V)}_{(A)\nu}(p,q,e) \\ \chi^{(T)}_{\mu\nu}(\Lambda p,\Lambda q,\Lambda e) &= \Lambda_{\mu\lambda}\Lambda_{\nu\sigma}\chi^{(T)}_{\lambda\sigma}(p,q,e) . \end{aligned} \tag{I-F-2}$$

Under parity transformations, eq. (I-E-13) implies that if $\eta_P^{(B)}\eta_P^{(1)}\eta_P^{(2)} = (-1)^j$, then

$\chi^{(S)}$ = scalar function

$\chi^{(P)}$ = pseudoscalar function

$\chi^{(V)}$ = vector function

$\chi^{(A)}$ = pseudovector function

$\chi^{(T)}$ = tensor function.

If $\eta_P^{(B)}\eta_P^{(1)}\eta_P^{(2)} = -(-1)^j$, then the reverse holds.

Since the bound state is linear in the polarization tensor e, the B-S wavefunction must also be linear in e. Using this fact, one can construct tensor functions of q, p, and e with the correct parity. For j=0 there are four such functions. For $j > 0$ there are eight.

For $j > 0$, define

$$q \cdot e \equiv q_{\mu_1} q_{\mu_2} \cdots q_{\mu_j} e_{\mu_1 \mu_2 \cdots \mu_j} \tag{I-F-3}$$

$$\tilde{e}_\mu \equiv q_{\mu_1} q_{\mu_2} \cdots q_{\mu_{j-1}} e_{\mu \mu_1 \mu_2 \cdots \mu_{j-1}} . \tag{I-F-4}$$

Then for $\eta_P^{(B)} \eta_P^{(1)} \eta_P^{(2)} = (-1)^j$, the wavefunction can be written as

$$\begin{aligned} \chi(p,q,e) = {} & i\chi^{(1)} q \cdot e - i\chi^{(2)} q \cdot e\, \gamma \cdot p \\ & + \chi^{(3)} q \cdot e\, \gamma \cdot q + \chi^{(4)} q \cdot e\, p_\mu q_\nu \sigma_{\mu\nu} \\ & + \chi^{(5)} \gamma \cdot \tilde{e} + \chi^{(6)} \epsilon_{\mu\nu\lambda\sigma} \tilde{e}_\mu p_\nu q_\lambda \gamma_\sigma \gamma_5 \\ & + \chi^{(7)} \tilde{e}_\mu p_\nu \sigma_{\mu\nu} + i\chi^{(8)} \tilde{e}_\mu q_\nu \sigma_{\mu\nu} , \end{aligned} \tag{I-F-5}$$

where

$$\chi^{(i)} \equiv \chi^{(i)}(q^2, p \cdot q)$$

is a Lorentz invariant function. The factors of i and -1 are chosen to simplify the problem after Wick rotation. Using time reversal and parity symmetries, as expressed in eq. (I-E-23), one can derive the following reality conditions on the invariant amplitides:

$$\chi^{(i)}(q^2, p \cdot q) = \begin{cases} \eta\, \chi^{(i)*}(\vec{q}^{\,2} - q_0^{*2}, \vec{p} \cdot \vec{q} - p_0 q_0^*) & \begin{pmatrix} i = 1,3,4, \\ 5,6,7 \end{pmatrix} \\ -\eta\, \chi^{(i)*}(\vec{q}^{\,2} - q_0^{*2}, \vec{p} \cdot \vec{q} - p_0 q_0^*) & (i = 2,8) . \end{cases} \tag{I-F-6}$$

The phase factor η is arbitrary, but is equal for all eight amplitudes. If one carries out the Wick rotation in the rest frame ($\vec{p} = 0$) of the bound state, then this relation specializes to

$$\chi^{(i)}(q^2, p\cdot q) = \begin{cases} \eta\,\chi^{(i)*}(q^2, -p\cdot q) & \begin{pmatrix} i = 1,3,4, \\ 5,6,7 \end{pmatrix} \\ -\eta\,\chi^{(i)*}(q^2, -p\cdot q) & (i = 2,8). \end{cases} \qquad \text{(I-F-7)}$$

If the bound state has a definite charge conjugation number, then eq. (I-E-17) implies that

$$\chi^{(i)}(q^2, -p\cdot q) = \begin{cases} (-1)^j\,\eta_C^{(B)}\chi^{(i)}(q^2, p\cdot q) & \begin{pmatrix} i = 1,3,4, \\ 5,6,7 \end{pmatrix} \\ -(-1)^j\,\eta_C^{(B)}\chi^{(i)}(q^2, p\cdot q) & (i = 2,8). \end{cases} \qquad \text{(I-F-8)}$$

By combining the previous two equations, one sees that if charge conjugation applies and parity and time reversal symmetries hold, then all the amplitudes may be taken as real in the Wick rotated region. This result can also be obtained more directly from eq. (I-E-24), which is a statement of CT.

For j = 0, the most general expansion is given by the first four terms of eq. (I-F-5), with the factor q.e omitted. Eqs. (I-F-6) - (I-F-8) hold for this case, too.

For $\eta_P{}^{(B)}\eta_P{}^{(1)}\eta_P{}^{(2)} = -(-1)^j$, the corresponding equations are

$$\begin{aligned} \chi(p,q,e) = {} & \chi^{(1)} q\cdot e\,\gamma - i\,\chi^{(2)} q\cdot e\,\gamma\cdot p\,\gamma_5 \\ & + \chi^{(3)} q\cdot e\,\gamma\cdot q\,\gamma_5 - i\,\chi^{(4)} q\cdot e\,\epsilon_{\mu\nu\lambda\sigma} p_\mu q_\nu \sigma_{\lambda\sigma} \\ & + \chi^{(5)} \epsilon_{\mu\nu\lambda\sigma} \tilde{e}_\mu p_\nu q_\lambda \gamma_\sigma + \chi^{(6)} \gamma\cdot\tilde{e}\,\gamma_5 \\ & - i\,\chi^{(7)} \epsilon_{\mu\nu\lambda\sigma} \tilde{e}_\mu p_\nu \sigma_{\lambda\sigma} + \chi^{(8)} \epsilon_{\mu\nu\lambda\sigma} \tilde{e}_\mu q_\nu \sigma_{\lambda\sigma} \end{aligned} \qquad \text{(I-F-5')}$$

$$\chi^{(i)}(q^2, p\cdot q) = \begin{cases} \eta\, \chi^{(i)*}(\vec{q}^{\,2} - q_0^{*2}, \vec{p}\cdot\vec{q} - p_0 q_0^*) & \binom{i=1,2,}{4,7} \\ -\eta\, \chi^{(i)*}(\vec{q}^{\,2} - q_0^{*2}, \vec{p}\cdot\vec{q} - p_0 q_0^*) & \binom{i=3,5,}{6,8} \end{cases} \tag{I-F-6'}$$

$$\chi^{(i)}(q^2, p\cdot q) = \begin{cases} \eta\, \chi^{(i)*}(q^2, -p\cdot q) & \binom{i=1,2,}{4,7} \\ -\eta\, \chi^{(i)*}(q^2, -p\cdot q) & \binom{i=3,5,}{6,8} \end{cases} \tag{I-F-7'}$$

$$\chi^{(i)}(q^2, -p\cdot q) = \begin{cases} (-1)^j\, \eta_C^{(B)} \chi^{(i)}(q^2, p\cdot q) & \binom{i=1,2,}{4,7} \\ -(-1)^j\, \eta_C^{(B)} \chi^{(i)}(q^2, p q) & \binom{i=3,5,}{6,8} \end{cases} \tag{I-F-8'}$$

All amplitudes may be taken as real under the same conditions as before. The correct equations for j = 0 are also obtained as before.

G. RELATION OF THE BETHE-SALPETER WAVEFUNCTION TO THE RESIDUE OF A POLE IN THE GREEN'S FUNCTION

Consider a matrix element of the form

$$M_1(P) \equiv \int d^4x_1 d^4x_2 \, e^{-i(q+P)\cdot x_1} e^{i(q-P)\cdot x_2} \tag{I-G-1}$$

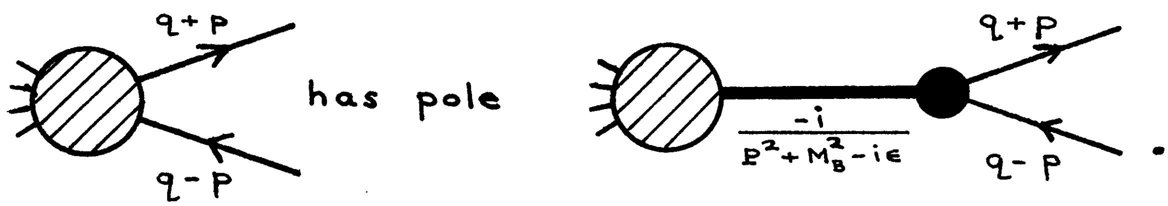
$$\times \langle 0 | T\{ \psi^{(1)}_\alpha(x_1)\, \overline{\psi}^{(2)}_\beta(x_2)\, A(y_1)\, B(y_2) \ldots X(y_n)\} | \alpha \rangle ,$$

where A,B,..., X, are any local operators and $|\alpha\rangle$ is any state. If there is a bound state with mass M_B and spin j, it will be shown that M(P) has a pole at $P^2 = -M_B^2 + i\epsilon$ of the form

$$M_1(P) \longrightarrow \sum_{\lambda=-j}^{j} \chi_{\alpha\beta}(P,q,\lambda) \frac{-i}{P^2 + M_B^2 - i\epsilon} \tag{I-G-2}$$

$$\times (2\pi)^{3/2} \langle P,\lambda | T\{ A(y_1)\, B(y_2) \ldots X(y_n)\} | \alpha \rangle .$$

Diagrammatically,



Similarly one can consider a matrix element of the form

$$M_2(P) \equiv \int d^4x_1 d^4x_2 \, e^{i(q+P)\cdot x_1} e^{-i(q-P)\cdot x_2} \tag{I-G-3}$$

$$\times \langle \alpha | T\{ A(y_1)\, B(y_2) \ldots X(y_n)\, \overline{\psi}^{(1)}_\beta(x_1)\, \psi^{(2)}_\alpha(x_2)\} | 0 \rangle .$$

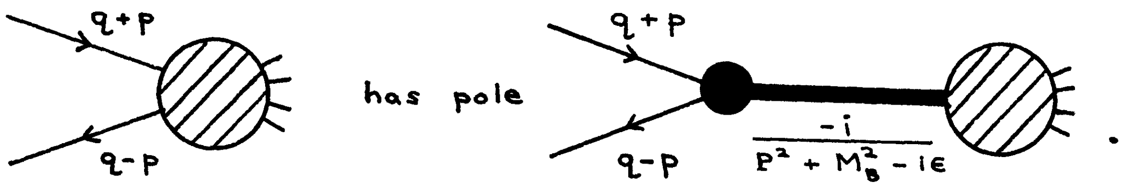
The bound state pole in $M_2(P)$ has the form

$$M_2(P) \longrightarrow \sum_{\lambda=-j}^{j} (2\pi)^{3/2} \langle \alpha | T\{ A(y_1)\, B(y_2)$$

$$\ldots X(y_n)\} | P,\lambda \rangle \times \tag{I-G-4}$$

$$\times \frac{-i}{P^2 + M_B^2 - i\epsilon} \quad \overline{\chi}_{\alpha\beta}(p, q, \lambda).$$

Diagrammatically,



These two statements are derived similarly, so we will sketch the derivation of only the first. The derivation follows a proof of the LSZ theorem given by Weinberg[10]. If we set

$$t_{min} = max\,(y_{10}, y_{20}, \ldots, y_{n0}), \qquad \text{(I-G-5)}$$

then for the range of integration $x_{10} > t_{min}$ and $x_{20} > t_{min}$, the $\psi\overline{\psi}$ in the expression for $M_1(P)$ may be taken to the left of the time ordered product. It is this part of the integration range which is solely responsible for the pole at $P^2 = -M_B^2$. A complete set of intermediate states can then be inserted between the $\psi\overline{\psi}$ and the time ordered product. Only the states of mass M_B contribute to the pole. Thus,

$$M\,(P) = \int_{t_{min}}^{\infty} d^4x_1 d^4x_2 e^{-i(q+p)\cdot x_1}\, e^{i(q-p)\cdot x_2} \qquad \text{(I-G-6)}$$

$$\times \sum_{\lambda} \int \frac{d^3P'}{2P'_0} \langle 0 | T\{\psi_\alpha^{(1)}(x_1)\, \overline{\psi}_\beta^{(2)}(x_2)\} | P'\lambda \rangle$$

$$\times \langle P'\lambda | T\{A(y_1)\, B(y_2) \ldots X(y_n)\} | \alpha \rangle$$

$$+ \text{ terms regular at } P^2 = -M_B^2,$$

where the limits of integration refer to the time components only. Now substitute

$$x_1' = x_1 - x_2 \qquad \text{(I-G-7)}$$

$$x_2' = \frac{x_1 + x_2}{2} ,$$

and use translation invariance to shift the arguments of the first matrix element.

$$M_1(P) \longrightarrow \sum_\lambda \int d^4x_1' \, e^{-iq\cdot x_1'} \int_{t_{min}}^{\infty} d^4x_2' \int \frac{d^3P'}{2P_0'} \, e^{ix_2'\cdot(P'-P)}$$
$$\times \langle 0 | T\{ \psi_\alpha^{(1)}(\tfrac{x_1'}{2}) \, \overline{\psi}_\beta^{(2)}(-\tfrac{x_1'}{2}) \} | P'\lambda \rangle \qquad \text{(I-G-8)}$$
$$\times \langle P'\lambda | T\{ A(y_1) B(y_2) \ldots X(y_n) \} | \alpha \rangle .$$

One can now extract the pole from an integral of the form

$$I = \int_{t_{min}}^{\infty} d^4x_2' \int \frac{d^3P'}{2P_0'} \, e^{ix_2'\cdot(P'-P)} f(P')$$
$$= \frac{i(2\pi)^3}{2P_0''} f(P'') \frac{e^{it_{min}(P_0 - P_0'')}}{P_0 - P_0'' + i\epsilon} , \qquad \text{(I-G-9)}$$

where $\qquad P'' = (\vec{P}, i\sqrt{\vec{P}^2 + M_B^2}) .$

The $i\epsilon$ must be inserted so that the integral will converge. At the pole $P_0 = P_0''$, which means that 1) the exponential in the numerator can be taken as unity; 2) $f(P'')$ can be taken as $f(P)$; and 3) the factor of $2P_0'$ in the denominator can be taken as $P_0 + P_0''$. This given a pole of the form

$$I \longrightarrow \frac{-i(2\pi)^3}{P^2 + M_B^2 - i\epsilon} f(P) . \qquad \text{(I-G-10)}$$

The pole in $M_1(P)$ can be written as

$$M(P) \longrightarrow \sum_{\lambda} \int d^4x_1' \, e^{-iq\cdot x_1'} \langle 0 | T\{ \psi_\alpha^{(1)}(\tfrac{x_1'}{2}) \, \overline{\psi}_\beta^{(2)}(-\tfrac{x_1'}{2}) \} | P\lambda \rangle$$

$$\times \frac{-i(2\pi)^3}{P^2 + M_B^2 - i\epsilon} \langle P\lambda | T\{ A(y_1)\, B(y_2) \ldots X(y_n) \} | \alpha \rangle . \tag{I-G-11}$$

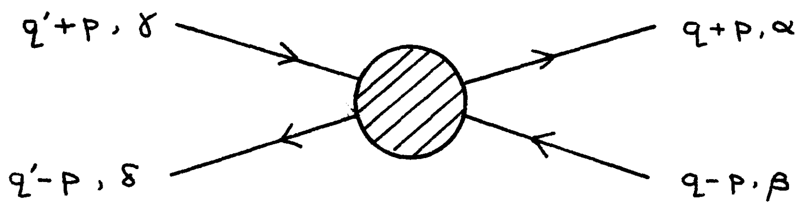
By referring to the definition of the B-S wavefunction (eq. I-B-1), one sees immediately that the result is proven.

H. BETHE-SALPETER EQUATION FOR THE FOUR-POINT GREEN'S FUNCTION

The four-point Green's function for fermion-antifermion scattering can be defined as

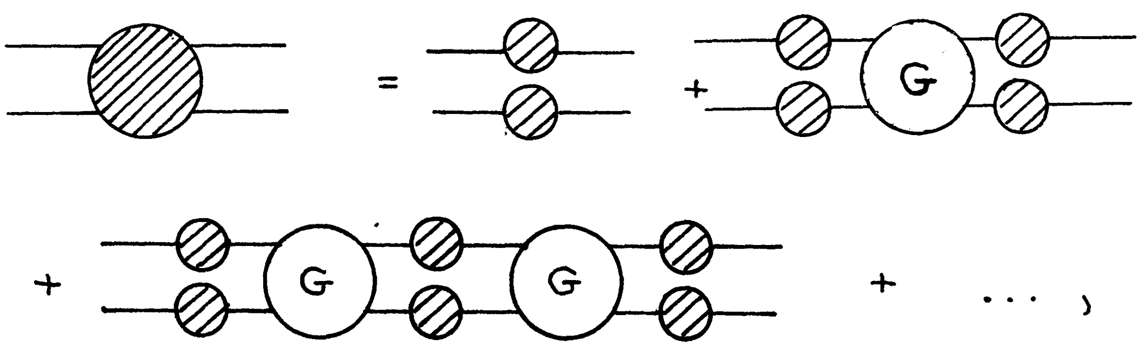
$$K_{\alpha\beta\gamma\delta}(p;q,q') \equiv \int d^4x_1\, d^4x_2\, d^4x_3\, e^{-i(q+p)\cdot x_1} e^{i(q-p)\cdot x_2}\, e^{i(q'+p)\cdot x_3} \langle 0|T\{\psi_\alpha^{(1)}(x_1)\, \bar{\psi}_\beta^{(2)}(x_2)\, \bar{\psi}_\gamma^{(1)}(x_3)\, \psi_\delta^{(2)}(0)\}|0\rangle ,$$

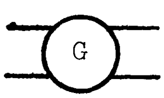
which can be shown diagrammatically as



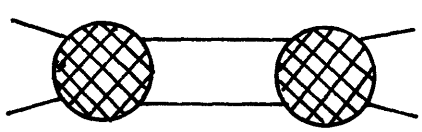
.

The function K can be expressed in perturbation theory as



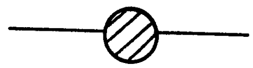
where 

represents the sum of all irreducible diagrams, without external legs. (A diagram is irreducible if it cannot be drawn in the form



.)

G will be referred to as the irreducible interaction function. The diagram



represents the full propagator. Algebraically,

$$K_{\alpha\beta\gamma\delta}(p;q,q') = K^{\circ}_{\alpha\beta\gamma\delta}(p;q,q')$$

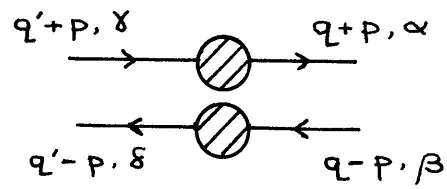
$$+ \int \frac{d^4\tilde{q}}{(2\pi)^4}\frac{d^4\tilde{q}'}{(2\pi)^4} K^{\circ}_{\alpha\beta\alpha'\beta'}(p;q,\tilde{q})\, G_{\alpha'\beta'\gamma'\delta'}(p;\tilde{q},\tilde{q}')$$

$$\times K^{\circ}_{\gamma'\delta'\gamma\delta}(p;\tilde{q}',q')$$

$$+ \quad \cdots ,$$

where

$$K^{\circ}_{\alpha\beta\gamma\delta}(p;q,q') \equiv$$



$$\equiv (2\pi)^4\,\delta^4(q'-q)\, S^{(1)}_{F\alpha\gamma}(q+p)\, S^{(2)}_{F\delta\beta}(q-p),$$

and

$$S^{(i)}_{F_{\alpha\beta}}(q) \equiv \int d^4x\, e^{-iq\cdot x} \langle 0|T\{\psi^{(i)}_{\alpha}(x)\,\bar{\psi}^{(i)}_{\beta}(0)\}|0\rangle.$$

It is convenient to adopt a matrix notation

$$K_{\alpha\beta\gamma\delta}(p;q,q') \longrightarrow K(p),$$

where the Dirac indices and the arguments q,q' are treated as suppressed indices which are summed over when matrices are multiplied. Explicitly, two matrices $A_{\alpha\beta\gamma\delta}(q,q')$ and $B_{\alpha\beta\gamma\delta}(q,q')$ are multiplied according to the rule

$$[AB]_{\alpha\beta\gamma\delta}(q,q') \equiv \int \frac{d^4\tilde{q}}{(2\pi)^4} A_{\alpha\beta\mu\nu}(q,\tilde{q}) B_{\mu\nu\gamma\delta}(\tilde{q},q').$$

Using this definition, eq. (I-H-2) can be written simply as

$$K(p) = K^{\circ}(p) + K^{\circ}(p)G(p)K^{\circ}(p)$$
$$+ K^{\circ}(p)G(p)K^{\circ}(p)G(p)K^{\circ}(p) + \ldots \; .$$

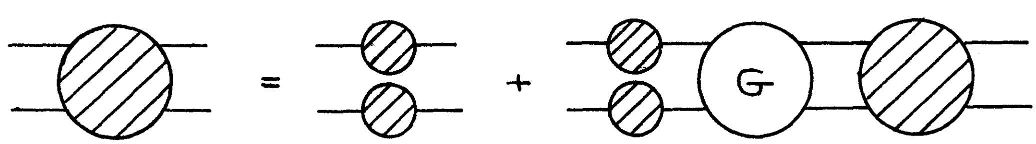
From the previous equation, it is clear that K(p) obeys the integral equations

$$K(p) = K^{\circ}(p) + K^{\circ}(p)G(p)K(p)$$

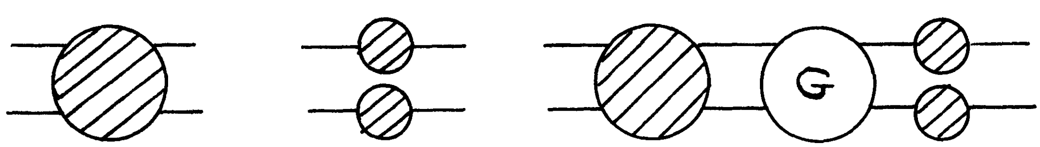
and

$$K(p) = K^{\circ}(p) + K(p)G(p)K^{\circ}(p).$$

This is the B-S equation for the four-point Green's function. Diagrammatically,



and



.

In this thesis we will be interested in three types of interactions -- scalar, pseudoscalar, and neutral vector exchange. For the case of neutral vector exchange, we assume that the vector field is coupled to a conserved vector current, so that only the $\delta_{\mu\nu}$ term of the spin one propagator will contribute to the S-matrix. The interaction Hamiltonian densities for these three interactions are, respectively,

$$\mathcal{H}_I^S = g\,\bar{\psi}(x)\,\psi(x)\,\phi(x)$$

$$\mathcal{H}_I^P = i g\,\bar{\psi}(x)\,\gamma_5\,\psi(x)\,\phi(x)$$

$$\mathcal{H}_I^V = i g\,\bar{\psi}(x)\,\gamma_\mu\,\psi(x)\,V_\mu(x)$$

The factors of i are added to make $\mathcal{H}$ Hermitean when g is real. In lowest order perturbation theory (known as the ladder approximation, or the single particle exchange approximation), the corresponding irreducible interaction functions are

$$G_{\alpha\beta\gamma\delta}(p,q,q') = \begin{cases} \dfrac{i g^2}{(q-q')^2+\mu^2-i\epsilon}\,\delta_{\alpha\gamma}\,\delta_{\delta\beta} & (S) \\[2ex] \dfrac{-i g^2}{(q-q')^2+\mu^2-i\epsilon}\,\gamma_{5\alpha\gamma}\,\gamma_{5\delta\beta} & (P) \\[2ex] \dfrac{-i g^2}{(q-q')^2+\mu^2-i\epsilon}\,\gamma_{\mu_{\alpha\gamma}}\,\gamma_{\mu_{\delta\beta}} & (V) \end{cases}$$

where μ is the mass of the exchanged particle. The value for the propagator S_F is given in lowest order perturbation theory as

$$S^{(i)}_{F\alpha\beta}(q) = i\,\frac{i\gamma\cdot q - m_i}{q^2 + m_i^2 - i\epsilon} ,$$

where m_i is the mass of the particle. So in this approximation

$$K^{o}_{\alpha\beta\gamma\delta}(p;q,q') = -(2\pi)^4\,\delta^4(q'-q)$$

$$\frac{[i\gamma\cdot(q+p) + m_1]_{\alpha\gamma}\,[i\gamma\cdot(q-p) - m_2]_{\delta\beta}}{[(q+p)^2 + m_1^2 - i\epsilon]\,[(q-p)^2 + m_2^2 - i\epsilon]} .$$

I. BETHE-SALPETER EQUATION FOR THE BOUND STATE WAVEFUNCTION

As shown in section I-G, the Green's function has bound state poles with residues proportional to the B-S wavefunction. Using the results of that section, one can show that the four point Green's function contains the poles

$$K_{\alpha\beta\gamma\delta}(P,q,q') \xrightarrow[P^2\to -M_B^2]{} \sum_\lambda \chi_{\alpha\beta}(P,q,\lambda) \times \frac{-i}{P^2+M_B^2-i\epsilon}\,\overline{\chi}_{\delta\gamma}(P,q',\lambda)\,, \tag{I-I-1}$$

where we have expressed everything in terms of 2P 2P. Using the matrix notation, this can be written as

$$K(P) \xrightarrow[P^2\to -M_B^2]{} \sum_\lambda \chi(P,\lambda)\frac{-i}{P^2+M_B^2-i\epsilon}\,\overline{\chi}^T(P,\lambda), \tag{I-I-2}$$

where χ and $\overline{\chi}$ are regarded as vectors with suppressed Dirac indices and a suppressed index q. The superscript T on $\overline{\chi}$ indicates that the Dirac indices are transposed.

By equating the residues of the pole at $P^2 = -M_B^2$ of both sides of the B-S equations for the four-point Green's function (eqs. I-H-7 and 8), one arrives at

$$\sum_\lambda \chi(P_B,\lambda)\,\overline{\chi}^T(P_B,\lambda) = \sum_\lambda K^0(P_B)\,G(P_B)\,\chi(P_B,\lambda)\,\overline{\chi}^T(P_B,\lambda)$$

$$\sum_\lambda \chi(P_B,\lambda)\,\overline{\chi}^T(P_B,\lambda) = \sum_\lambda \chi(P_B,\lambda)\,\overline{\chi}^T(P_B,\lambda)\,G(P_B)\,K^0(P_B) \tag{I-I-3}$$

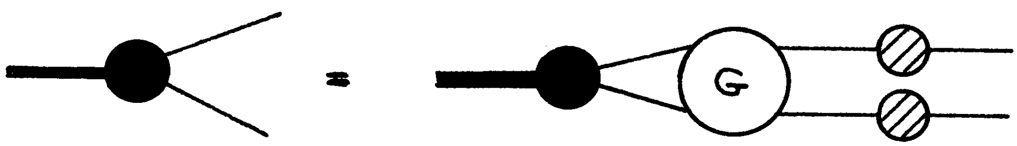
where P_B is any four-vector satisfying

$$P_B^2 = -M_B^2\,. \tag{I-I-4}$$

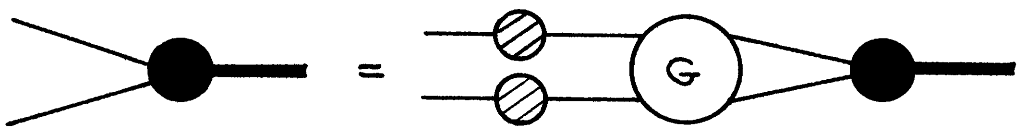
The $\chi(P,\lambda)$, for $\lambda = -j,\ldots,j$, are known to be linearly independent since they transform under rotations according to the (2j + 1) - dimensional irreducible representation of the rotation group. It then follows from the above equations that

$$\chi(P_B,\lambda) = K^0(P_B)\,G(P_B)\,\chi(P_B,\lambda),$$
$$\bar{\chi}^T(P_B,\lambda) = \bar{\chi}^T(P_B,\lambda)\,G(P_B)\,K^0(P_B). \qquad \text{(I-I-5)}$$

This is the B-S equation for the bound state wavefunction. Diagrammatically,



and



.

In ladder approximation, these equations can be written explicitly by using eq. (I-H-10) for G and (I-H-12) for K^0. For a scalar interaction,

$$\chi(p,q,\lambda) = -ig^2\left[\frac{i\gamma\cdot(q+p)-m_1}{(q+p)^2+m_1^2-i\epsilon}\right]\int\frac{d^4K}{(2\pi)^4}\,\frac{\chi(p,K,\lambda)}{(q-K)^2+\mu^2-i\epsilon}$$
$$\times\left[\frac{i\gamma\cdot(q-p)-m_2}{(q-p)^2+m_2^2-i\epsilon}\right]. \qquad \text{I-I-6)}$$

For a pseudoscalar interaction,

$$\chi(p,q,\lambda) = -ig^2\left[\frac{i\gamma\cdot(q+p)-m_1}{(q+p)^2+m_1^2-i\epsilon}\right]\int\frac{d^4K}{(2\pi)^4}\,\frac{\gamma_5\,\chi(p,K,\lambda)\,\gamma_5}{(q-K)^2+\mu^2-i\epsilon}$$
$$\times\left[\frac{i\gamma\cdot(q-p)-m_2}{(q-p)^2+m_2^2-i\epsilon}\right]. \qquad \text{(I-I-7)}$$

And for a neutral vector interaction,

$$\chi(p,q,\lambda) = -ig^2\left[\frac{i\gamma\cdot(q+p)-m_1}{(q+p)^2+m_1^2-i\epsilon}\right]\int\frac{d^4K}{(2\pi)^4}\,\frac{\gamma_\mu\,\chi(p,K,\lambda)\,\gamma_\mu}{(q-K)^2+\mu^2-i\epsilon}\,\times$$

$$\times \left[\frac{i\gamma\cdot(q-p) - m_2}{(q-p)^2 + m_2^2 - i\epsilon} \right]. \qquad \text{(I-I-8)}$$

These equations can be Wick rotated provided that $\chi(p,k,\lambda)$ falls off fast enough for large k_o so that the contour at infinity can be neglected. The factor of $[(q-k)^2 + \mu^2 - i\epsilon]^{-1}$ in the integrand will place no singularities in the path of the contour of integration, provided that q and k are both Wick rotated together. For a scalar interaction, the B-S equation becomes, after Wick rotation,

$$\chi(\bar{p},\bar{q},\lambda) = g^2 \left[\frac{\gamma\cdot(i\bar{q}-\bar{p}) - m_1}{(\bar{q}+i\bar{p})^2 + m_1^2} \right] \int \frac{d^4\bar{k}}{(2\pi)^4} \frac{\chi(\bar{p},\bar{k},\lambda)}{(\bar{q}-\bar{k})^2 + \mu^2} \qquad \text{(I-I-9)}$$

$$\times \left[\frac{\gamma\cdot(i\bar{q}+\bar{p}) - m_2}{(\bar{q}-i\bar{p})^2 + m_2^2} \right].$$

If $m_1 = m_2 = m$, the denominator can be simplified to give

$$\chi(\bar{p},\bar{q},\lambda) = \frac{g^2}{(\bar{q}^2 - \bar{p}^2 + m^2)^2 + 4(\bar{p}\cdot\bar{q})^2} [\gamma\cdot(i\bar{q}-\bar{p}) - m] \qquad \text{(I-I-10)}$$

$$\times \int \frac{d^4k}{(2\pi)^4} \frac{\chi(\bar{p},\bar{k},\lambda)}{(\bar{q}-\bar{k})^2 + \mu^2} [\gamma\cdot(i\bar{q}+\bar{p}) - m].$$

For a pseudoscalar interaction, the above equation is modified by the replacement

$$\chi(\bar{p},\bar{k},\lambda) \longrightarrow -\gamma_5\,\chi(\bar{p},\bar{k},\lambda)\,\gamma_5. \qquad \text{(I-I-11)}$$

For a neutral vector interaction, the modification is given by

$$\chi(\bar{p},\bar{k},\lambda) \longrightarrow -\gamma_\mu\,\chi(\bar{p},\bar{k},\lambda)\,\gamma_\mu. \qquad \text{(I-I-12)}$$

J. NORMALIZATION CONDITION FOR THE BETHE-SALPETER WAVEFUNCTION

The normalization condition for the B-S wavefunction is also determined by the B-S equation for the four-point Green's function. It is convenient to define the inverse of K^{O},

$$H(P) = K^{o}(P)^{-1}, \tag{I-J-1}$$

where the word "inverse" is to be interpreted using our definition of matrix multiplication. Explicitly,

$$H_{\alpha\beta\gamma\delta}(P;q,q') = (2\pi)^4 \delta^4(q'-q) S^{(1)-1}_{F_{\alpha\gamma}}(q+P) S^{(2)-1}_{F_{\delta\beta}}(q-P). \tag{I-J-2}$$

In terms of H, the B-S equation for the four-point Green's function (eq. I-H-7) becomes

$$[H(P) - G(P)]K(P) = 1. \tag{I-J-3}$$

The equation for $\overline{\chi}(P,\lambda)$ (eq. I-I-5) can be written

$$\overline{\chi}^{T}(P_B,\lambda)\,[H(P_B) - G(P_B)] = 0. \tag{I-J-4}$$

The final ingredient is the relation between χ and the bound state pole of K (eq.(I-I-2)), which can be written as

$$i \lim_{P^2\to -M_B^2} (P^2 + M_B^2) K(P) = \sum_{\lambda} \chi(P,\lambda)\overline{\chi}^{T}(P,\lambda). \tag{I-J-5}$$

If one multiplies eq. (I-J-3) by $(P^2 + M_B^2)$ and takes the limit as $P^2 \longrightarrow - M_B^2$, the result is the homogeneous B-S equation for $\chi(P)$. In order to obtain the normalization, one must look just off the pole. This may be done by multiplying eq. (I-J-3) by $(P^2 + M_B^2)$, then differentiating with respect to P_μ , and then taking the limit $P^2 \longrightarrow - M_B^2$.

$$\frac{\partial}{\partial P_\mu} \left\{ (P^2 + M_B^2) \left[H(P) - G(P) \right] K(P) \right\} = 2 P_\mu \qquad \text{(I-J-6)}$$

$$= (P^2 + M_B^2) \frac{\partial}{\partial P_\mu} \left[H(P) - G(P) \right] K(P) \qquad \text{(I-J-7)}$$

$$+ \left[H(P) - G(P) \right] \frac{\partial}{\partial P_\mu} \left[(P^2 + M_B^2) K(P) \right].$$

Now take the limit as $P^2 \longrightarrow - M_B^2$ (or equivalently, as P P_B), and multiply on the left by $\overline{\chi}^T(P_B, \lambda')$.

$$\overline{\chi}^T(P_B, \lambda') \lim_{P \to P_B} \frac{\partial}{\partial P_\mu} \left\{ (P^2 + M_B^2) \left[H(P) - G(P) \right] K(P) \right\} \qquad \text{(I-J-8)}$$

$$= \quad 2 P_{B\mu} \, \overline{\chi}^T(P_B, \lambda')$$

$$= -i \, \overline{\chi}^T(P_B, \lambda') \frac{\partial}{\partial P_{B\mu}} \left[H(P_B) - G(P_B) \right] \sum_\lambda \chi(P_B, \lambda) \, \overline{\chi}^T(P_B, \lambda). \qquad \text{(I-J-9)}$$

Once again relying on the linear independence of the B-S wavefunctions for different values of λ, it follows that the B-S wavefunction obeys the following normalization condition:

$$\overline{\chi}(P_B, \lambda') \frac{\partial}{\partial P_{B\mu}} \left[H(P_B) - G(P_B) \right] \chi(P_B, \lambda) \qquad \text{(I-J-10)}$$

$$= 2i \, \delta_{\lambda' \lambda} P_\mu .$$

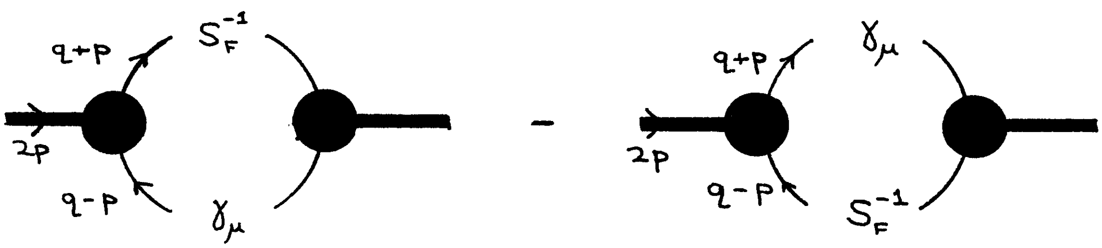
The above formula takes on a very simple form in lowest order perturbation theory. In ladder approximation (eq. (I-H-10)),

$$\frac{\partial G(P)}{\partial P_\mu} = 0, \qquad \text{(I-J-11)}$$

so only the inverse propagator contributes. Taking the lowest order propagator (eq. (I-H-11)), the normalization condition becomes

$$\begin{aligned}\int \frac{d^4q}{(2\pi)^4} \mathrm{Tr}\Big\{ &\overline{\chi}(P,q,\lambda')\, i[i\gamma\cdot(q+P)+m_1]\,\chi(P,q,\lambda)\,\gamma_\mu \\ &-\overline{\chi}(P,q,\lambda')\,\gamma_\mu\,\chi(P,q,\lambda)\, i[i\gamma\cdot(q-P)+m_2]\Big\} \\ &= 4i\,\delta_{\lambda'\lambda}\,P_\mu .\end{aligned} \qquad \text{(I-J-12)}$$

Diagrammatically,



$$= 4i\,P_\mu .$$

As will be seen in the next section, this form of the normalization condition is equivalent to the calculation of matrix elements of a conserved vector current.

If $m_1 = m_2 = m$ and the bound state has a definite charge conjugation number, then eq. (I-E-17) can be used to show that the two terms on the left-hand side of eq. (I-J-12) give equal contributions. Wick rotating, the normalization condition then takes the form

$$\begin{aligned}\int \frac{d^4\bar{q}}{(2\pi)^4} \mathrm{Tr}\left\{ \overline{\chi}(\bar{P},\bar{q},\lambda')\,\gamma_\mu\,\chi(\bar{P},\bar{q},\lambda)\,[\gamma\cdot(i\bar{q}+\bar{P})+m]\right\} \\ = -4\bar{P}_\mu\,\delta_{\lambda'\lambda} .\end{aligned} \qquad \text{(I-J-13)}$$

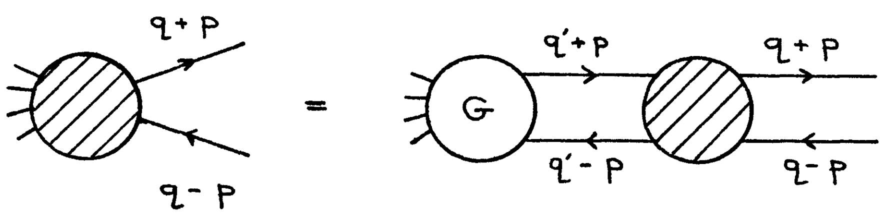
K. CALCULATION OF MATRIX ELEMENTS INVOLVING BOUND STATES

Suppose one wishes to compute a matrix element of the form

$$\mathcal{M}_1(P,\lambda) = \langle P\lambda | T\{A(y_1)\, B(y_2) \ldots X(y_n)\} | \alpha \rangle, \qquad \text{(I-K-1)}$$

where the notation is that of section I-G. Eq. (I-G-2) shows how this matrix element can be extracted as the residue of the bound state pole of the matrix element $M_1(P)$, defined in eq. (I-G-1).

In perturbation theory, $M_1(P)$ can be expressed as an infinite sum of Feynman diagrams. These diagrams can be grouped so that $M_1(P)$ can be written as



,

where G is the sum of all diagrams which are irreducible with respect to the two lines on the right, and with no external legs for these lines. The blob on the right is the four-point Green's function. Algebraically,

$$M_{1_{\alpha\beta}}(P) = \int \frac{d^4q'}{(2\pi)^4}\, K_{\alpha\beta\gamma\delta}(P;q,q') \times G_{\gamma\delta}(P,q',y\,,y\,,\cdots y\,)\,. \qquad \text{(I-K-2)}$$

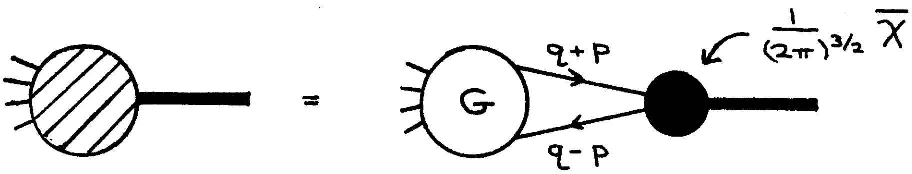
Now multiply both sides of this equation by $i(P^2 + M_B^2)$ and take the limit as $P^2 \longrightarrow -M_B^2$. Use eq. (I-G-2) to evaluate the left-hand side, and eq. (I-I-1) to evaluate the right-hand side.

$$\sum_{\lambda} \chi_{\alpha\beta}(p,q,\lambda)\,(2\pi)^{3/2}\,\mathcal{M}_1(P,\lambda)$$
$$= \int \frac{d^4q'}{(2\pi)^4} \sum_{\lambda} \chi_{\alpha\beta}(p,q,\lambda)\,\overline{\chi}_{\delta\gamma}(p,q',\lambda) \times G_{\gamma\delta}(p,q',y_1,y_2,\ldots,y_n)\,. \tag{I-K-3}$$

Due to the linear independence of the wavefunctions, it follows that

$$\mathcal{M}_1(P,\lambda) = \int \frac{d^4q'}{(2\pi)^4}\,\frac{1}{(2\pi)^{3/2}}\,\overline{\chi}_{\delta\gamma}(p,q',\lambda) \times G_{\gamma\delta}(p,q',y_1,y_2,\ldots,y_n)\,. \tag{I-K-4}$$

Diagrammatically,



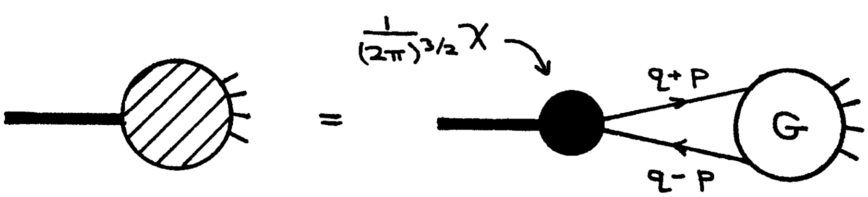
Similarly, a matrix element of the form

$$\mathcal{M}_2(P,\lambda) \equiv \langle \alpha \,|\, T\{A(y_1)\,B(y_2)\ldots X(y_n)\}\,|\, P\,\lambda \rangle \tag{I-K-5}$$

can be expressed as

$$\mathcal{M}_2(P,\lambda) = \int \frac{d^4q'}{(2\pi)^4}\,G_{\gamma\delta}(y_1,y_2,\ldots,y_n,p,q') \times \frac{1}{(2\pi)^{3/2}}\,\chi_{\gamma\delta}(p,q',\lambda)\,. \tag{I-K-6}$$

Diagrammatically,



In this thesis we will be interested in three types of decays:

$$P \longrightarrow \ell \nu$$

$$V \longrightarrow \ell^+ \ell^-$$

$$V \longrightarrow P \gamma .$$

where P represents a pseudoscalar meson, V a vector meson, ℓ a lepton (e or μ), and ν a neutrino. We will also want to calculate the magnetic moment of the vector mesons. These four calculations will be carried out in the next four sections.

L. WEAK LEPTONIC DECAY OF THE PSEUDOSCALAR MESONS ($P \longrightarrow \ell\nu$)

We assume that the weak interactions can be described in lowest order perturbation theory by an effective current-current interaction:

$$\mathcal{H}^{eff}_{int}(x) = \frac{G}{\sqrt{2}}\left[J^{(\ell)}_{\sigma}(x)\, J^{(h)+}_{\sigma}(x) + h.c.\right], \qquad \text{(I-L-1)}$$

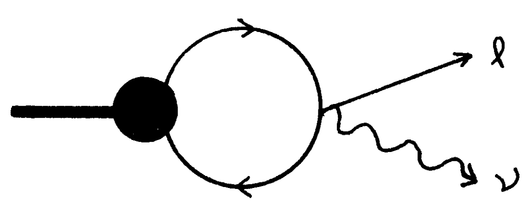
where

$$J^{(\ell)}_{\sigma}(x) = i\bar{\psi}_{\nu_e}(x)\gamma_\sigma(1+\gamma_5)\psi_e(x) + i\bar{\psi}_{\nu_\mu}(x)\gamma_\sigma(1+\gamma_5)\psi_\mu(x), \qquad \text{(I-L-2)}$$

and

$$J^{(h)}_{\sigma}(x) = i\bar{\psi}_p(x)\gamma_\sigma(1+\gamma_5)\left[\psi_n(x)\cos\theta_C + \psi_\lambda(x)\sin\theta_C\right], \qquad \text{(I-L-3)}$$

where p,n, and λ are quark indices and θ_C is the Cabbibo angle. The decay is then given by the diagram



One can define the constants f_π and f_K by the relations

$$\langle 0|J^{(h)+}_{\sigma}(x)|\pi^+(P)\rangle = i\frac{e^{iP\cdot x}}{(2\pi)^{3/2}} f_\pi \cos\theta_C\, P_\sigma \qquad \text{(I-L-4)}$$

and

$$\langle 0 | J_\sigma^{(h)+}(x) | K^+(P) \rangle = i \frac{e^{iP\cdot x}}{(2\pi)^{3/2}} f_K \sin\theta_C \, P_\sigma . \quad \text{(I-L-5)}$$

In terms of these constants, the decay rates are calculated to be

$$\Gamma_{\pi\to\mu\nu} = \frac{G^2 m_\mu^2}{8\pi m_\pi^3} (m_\pi^2 - m_\mu^2) f_\pi^2 \cos^2\theta_C \qquad \text{(I-L-6)}$$

and

$$\Gamma_{K\to\mu\nu} = \frac{G^2 m_\mu^2}{8\pi m_K^3} (m_K^2 - m_\mu^2) f_K^2 \sin^2\theta_C . \qquad \text{(I-L-7)}$$

If θ_C is taken from a fit to hyperon decays (sin θ_C = .23), one finds that $f_\pi \approx$ 130 Mev, and $f_K \approx$ 150 Mev.

The value of f_π and f_K can be related to the B-S wavefunctions by means of the general techniques discussed above, but this case is so simple that the general formalism is unnecessary. The matrix elements on the left-hand side of eqs. (I-L-4 and 5) can be expressed in terms of the matrix element of the two fields between the bound state and the vacuum, and that is how the B-S wavefunction is defined. The result is simply

$$f_{\pi \atop K} = \frac{1}{M_B^2} \int \frac{d^4q}{(2\pi)^4} \mathrm{Tr} \{ \chi(p,q) \, \gamma\cdot P \, \gamma_5 \} . \qquad \text{(I-L-8)}$$

Note that this formula is true to all orders in the strong interactions, although it relies on a first order treatment of the weak interactions.

If the above equation is Wick rotated, the result is given by

$$f_{\pi \atop K} = -\frac{2}{M_B^2} \int \frac{d^4\bar{q}}{(2\pi)^4} \mathrm{Tr} \{ \chi(\bar{p},\bar{q}) \, \gamma\cdot\bar{p} \, \gamma_5 \} . \qquad \text{(I-L-9)}$$

M. ELECTROMAGNETIC LEPTONIC DECAY OF THE VECTOR MESONS ($V \longrightarrow \ell^+\ell^-$)

The electromagnetic interactions can be described by an interaction Hamiltonian density of

$$\mathcal{H}_{int}(x) = -j_\sigma(x) A_\sigma(x), \tag{I-M-1}$$

where

$$j_\sigma(x) = j_\sigma^{(\ell)}(x) + j_\sigma^{(h)}(x), \tag{I-M-2}$$

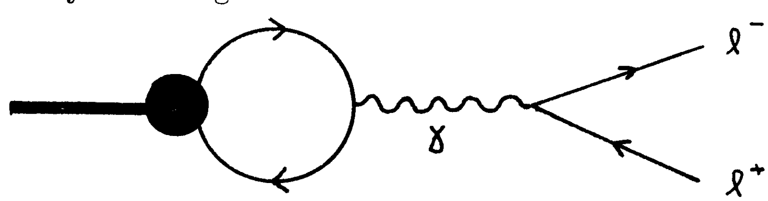
$$j_\sigma^{(\ell)}(x) = -ie\bar{\psi}_e(x)\gamma_\sigma\psi_e(x) - ie\bar{\psi}_\mu(x)\gamma_\sigma\psi_\mu(x), \tag{I-M-3}$$

$$j_\sigma^{(h)}(x) = \sum_{q=p,n,\lambda} ie_q\bar{\psi}^{(q)}(x)\gamma_\sigma\psi^{(q)}(x), \tag{I-M-4}$$

with

$$e_p = \frac{2}{3}e, \qquad e_n = e_\lambda = -\frac{1}{3}e. \tag{I-M-5}$$

Here e is the absolute value of the charge of an electron. The decay is then given by the diagram



The current matrix elements of the vector mesons can be parameterized using the constants g_V ($V = \rho, \omega, \phi, K^*$).

$$\langle 0 | j_\sigma(x) | P e \rangle = \frac{e^{iP\cdot x}}{(2\pi)^{3/2}} g_V M_B e_\sigma$$

Another popular parameterization uses a constant γ , related to g_V by

$$\gamma = \frac{e M_B}{2 g_V} .$$

In terms of the parameter g_V, the decay rate is given by

$$\Gamma_{V\to \ell^+\ell^-} = \frac{e^2 g_V}{12\pi} \frac{(M_B^2 + 2m_\ell^2)\sqrt{M_B^2 - 4m_\ell^2}}{M_B^4}$$

where m_ℓ is the mass of the lepton (e or μ). Neglecting the lepton mass,

$$\Gamma_{V\to \ell^+\ell^-} = \frac{\alpha}{3} \frac{g_V^2}{M_B} ,$$

where $\alpha = \frac{e^2}{4\pi} = \frac{1}{137}$.

The constant g_V is measured by either the $\mu^+\mu^-$ or e^+e^- decays, and the value should be equal. Following are the values of g_V calculated from the data compiled by the Particle Data Group[11]:

g_ρ (e^+e^-) = 49±5 Mev

g_ρ $(\mu^+\mu^-)$ = 51±12 Mev

g_ω (e^+e^-) = 15.5±2.2 Mev

(includes scale factor S = 1.4)

g_ϕ (e^+e^-) = 24.2±1.4 Mev

g_ϕ $(\mu^+\mu^-)$ = 19.6±2.3 Mev .

The experimental situation is somewhat confused, as evidenced by the rather large scale factor for g_ω (which indicates a disagreement of the different experiments) and the disagreement between the two values of g_ϕ.

To relate g_V to the B-S wavefunction, consider first a bound state of a pure quark-antiquark of the same type. Then g_V can be calculated by the same simple method used to calculate f_π and f_k.

$$g_V = \frac{ie_q}{M_B} \int \frac{d^4q}{(2\pi)^4} \mathrm{Tr} \{ \chi(p,q,e) \, \gamma \cdot e \} .$$

where e_q is the charge of the quark. This result is an exact consequence of eq. (I-M-4). Assuming "ideal" mixing, the quark content of the vector mesons can be taken as

$$|\rho^\circ\rangle = \frac{1}{\sqrt{2}} \left(|n\bar{n}\rangle - |p\bar{p}\rangle \right)$$

$$|\omega\rangle = \frac{1}{\sqrt{2}} \left(|n\bar{n}\rangle + |p\bar{p}\rangle \right)$$

$$|\phi\rangle = |\lambda\bar{\lambda}\rangle .$$

If isospin symmetry (SU(2)) is good, then the B-S wavefunctions for the $n\bar{n}$ and $p\bar{p}$ components of the ρ will be equal; and the same can be said for the ω. So each meson can be described by a single B-S wavefunction $\chi(p,q,e)$, which we will define to be normalized according to eq. (I-J-10). Then define

$$\bar{g}_V \equiv \frac{ie}{M_B} \int \frac{d^4q}{(2\pi)^4} \mathrm{Tr} \{ \chi(p,q,e) \, \gamma \cdot e \} .$$

Then

$$g_V = c_V \, \bar{g}_V ,$$

where

$$c_\rho = \frac{1}{\sqrt{2}} \left[\left(-\frac{1}{3}\right) - \left(\frac{2}{3}\right) \right] = -\frac{1}{\sqrt{2}}$$

$$c_\omega = \frac{1}{\sqrt{2}} \left[\left(-\frac{1}{3}\right) + \left(\frac{2}{3}\right) \right] = \frac{1}{3\sqrt{2}}$$

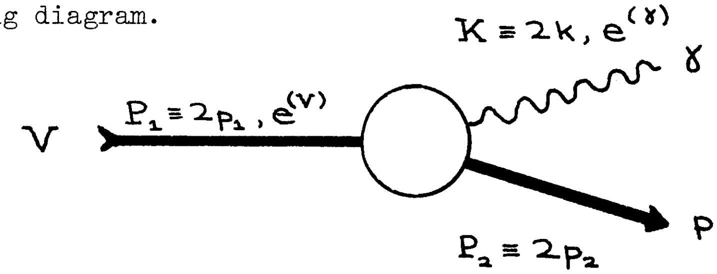
$$c_\phi = -\frac{1}{3} .$$

Wick rotating, the result is

$$\bar{g}_V = -\frac{e}{M_B} \int \frac{d^4\bar{q}}{(2\pi)^4} \operatorname{Tr} \left\{ \chi(\bar{p}, \bar{q}, e)\, \gamma \cdot e \right\} .$$

N. RADIATIVE DECAY OF THE VECTOR MESONS ($V \longrightarrow P\gamma$)

The kinematic variables for this decay are defined in the following diagram.



The S-matrix element for which we are looking is then

$$S_{fi} = \langle P(P_2)\gamma(K, e^{(\gamma)}) \,|\, S \,|\, V(P_1, e^{(V)})\rangle. \tag{I-N-1}$$

Using the reduction technique of LSZ[12], this matrix element can be expressed as

$$S_{fi} = i\,\frac{e_\mu^{(\gamma)*}}{(2\pi)^{3/2}} \langle P(P_2)\,|\,j_\mu(0)\,|\,V(P_1, e^{(V)})\rangle \times (2\pi)^4\,\delta^4(P_2 + K - P_1). \tag{I-N-2}$$

The above formula can also be derived by doing perturbation theory in the electromagnetic interaction.

The current matrix element can be parameterized by a single constant β_{VP}, defined by

$$\langle P(P_2)\,|\,j_\mu(0)\,|\,V(P_1, e^{(V)})\rangle = \frac{1}{(2\pi)^3}\,\beta_{VP}\,\epsilon_{\mu\nu\lambda\sigma}\,P_{2\nu}\,P_{1\lambda}\,e_\sigma^{(V)}. \tag{I-N-3}$$

The decay rate is then given by

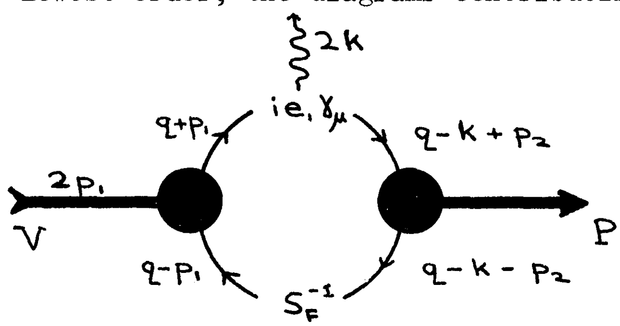
$$\Gamma_{V\to P\gamma} = \frac{1}{12\pi}\,\beta_{VP}^2\,t^3, \tag{I-N-4}$$

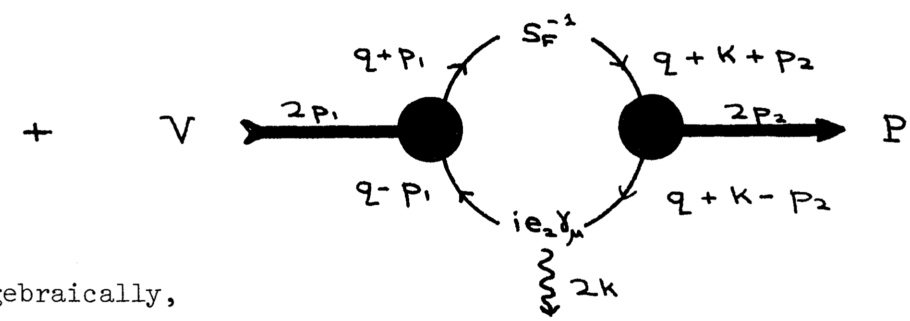
where

$$t = \frac{M_V^2 - M_P^2}{2M_V} \qquad \text{(I-N-5)}$$

is the magnitude of the three momentum of the P (or γ) in the rest frame of the V.

To calculate the matrix element of the current in terms of the B-S wavefunctions, one must use the full formalism developed in section I-K. Begin by assuming the bound state is composed of a quark of charge e_1 and an antiquark of charge $-e_2$. Superpositions will be considered later. In lowest order, the diagrams contributing are





Algebraically,

$$\langle P(P_2)|j_\mu(0)|V(P_1, e^{(V)}\rangle$$

$$= \frac{1}{(2\pi)^3}\int \frac{d^4q}{(2\pi)^4} \mathrm{Tr}\{\overline{\chi}^{(P)}(p_2, q-k)\, ie_1\gamma_\mu$$

$$\times \chi^{(V)}(p_1, q, e^{(V)})\, i[i\gamma\cdot(q-p_1) + m_2] +$$

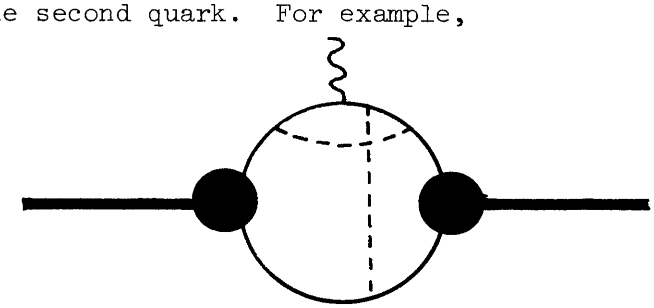
$$+ \bar{\chi}^{(P)}(p_2, q+k)\, i[i\gamma\cdot(q+p_1) + m_1]\, \chi^{(V)}(p_1, q, e^{(V)}) \times ie_2\gamma_\mu \Big\} . \tag{I-N-6}$$

We will be interested in cases where the states $|V\rangle$ and $|P\rangle$ have definite charge conjugation number -1 and +1, respectively. The B-S wavefunctions will then obey eq. (I-E-17), and under these circumstances the second term of eq. (I-N-6) can be shown to be of the same form as the first.

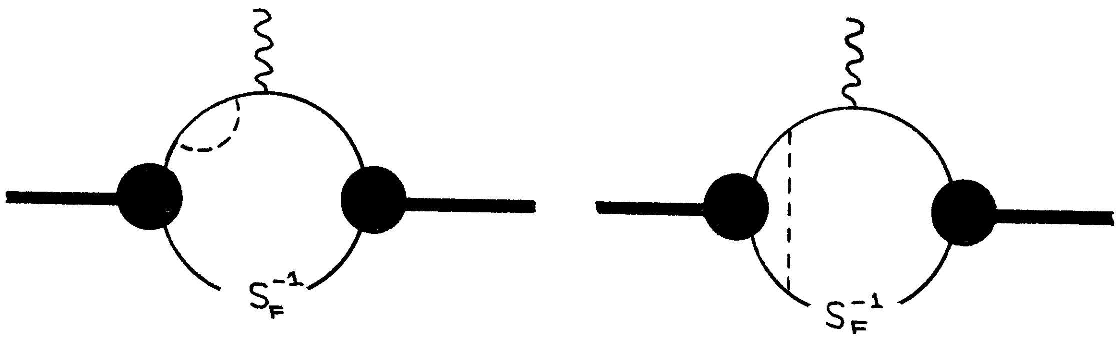
$$\langle P(P_2)|j_\mu(0)|V(P_1, e^{(V)})\rangle = \frac{1}{(2\pi)^3}\int \frac{d^4q}{(2\pi)^4} \times \mathrm{Tr}\Big\{\bar{\chi}^{(P)}(p_2, q-k)\, i(e_1+e_2)\gamma_\mu\, \chi^{(V)}(p_1, q, e^{(V)})\, i[i\gamma\cdot(q-p_1)+m]\Big\}, \tag{I-N-7}$$

where $m = m_1 = m_2$. Under the assumption of isospin invariance as well as charge conjugation, the above formula will also hold for $\rho^\pm \longrightarrow \pi^\pm\gamma$.

In contrast to the two previous calculations, this result is not true to all orders in the strong interactions. The inverse "bare" Feynman propagator should be replaced by the inverse of the full propagator. Graphs which correct the electromagnetic vertex are also allowed, so the γ_μ should be replaced by Γ_μ , the full vertex function. There are also graphs which may be regarded as the polarization of the electromagnetic vertex of one quark by the second quark. For example,



where the dashed line represents a strong interaction. The only excluded diagrams are those which modify the B-S wavefunctions by either self energy or vertex corrections:



(excluded)

An infinite number of diagrams contribute, and any explicit calculation must rely on the assumed dominance of a certain set of those diagrams.

In order to calculate β_{VP}, it is convenient to define $\bar{\beta}$ as the value that β_{VP} would have if the quark had charge e and the antiquark was uncharged. Then β_{VP} is determined by $\bar{\beta}$ and the quark content of V and P. Assuming that $|\rho\rangle$, $|\omega\rangle$, and $|\phi\rangle$ obey eqs. (I-M-11) ("ideal" mixing), and that the quark content of the following mesons is given by

$$\begin{aligned} |\rho^+\rangle &= |p\bar{n}\rangle \\ |\pi^+\rangle &= |p\bar{n}\rangle \\ |\pi^0\rangle &= \frac{1}{\sqrt{2}}\left[|p\bar{p}\rangle - |n\bar{n}\rangle\right] \\ |\eta\rangle &= \frac{1}{\sqrt{6}}\left[|p\bar{p}\rangle + |n\bar{n}\rangle - 2|\lambda\bar{\lambda}\rangle\right], \end{aligned} \tag{I-N-8}$$

then

$$\beta_{VP} = c_{VP}\,\bar{\beta}, \tag{I-N-9}$$

where

$$c_{\rho^0\pi^0} = \frac{1}{3}$$

$$C_{\omega\pi^\circ} = 1$$
$$C_{\phi\pi^\circ} = 0$$
$$C_{\rho^\pm\pi^\pm} = \frac{1}{3}$$
$$C_{\rho^\circ\eta} = \frac{1}{\sqrt{3}} \qquad \text{(I-N-10)}$$
$$C_{\omega\eta} = \frac{1}{3\sqrt{3}}$$
$$C_{\phi\eta} = \sqrt{2}/3 \; .$$

The only one of these reactions that has been measured is $\omega \longrightarrow \pi^\circ \gamma$[11], which gives a value $\bar{\beta} = (8.5 \pm 0.6) \times 10^{-4}$ Mev^{-1}.

The easiest way to calculate $\bar{\beta}$ is to calculate the current matrix element for a particular kinematic situation. Let

$$\begin{aligned} P_1 &= (0, 0, 0, iM_v) \\ e^{(v)} &= (0, 0, 1, 0) \\ P_2 &= (-t, 0, 0, i\sqrt{M_p^2 + t^2}\,) \\ K &= (t, 0, 0, it) \\ \mu &= 2 \end{aligned} \qquad \text{(I-N-11)}$$

Then

$$\begin{aligned} \bar{\beta} = \frac{i}{tM_v} \int \frac{d^4q}{(2\pi)^4} \, Tr \Big\{ \bar{\chi}^{(P)}(p_2, q-k) \, ie\Gamma_2 \\ \times \chi^{(v)}(p_1, q, e^{(v)}) \, i\,[i\gamma\cdot(q-p_1) + m] \Big\} . \end{aligned} \qquad \text{(I-N-12)}$$

It is in most cases possible to Wick rotate the above equation, but it is not clear that the Wick rotation simplifies the problem. The Wick rotation consists of a rotation of the contour of integration which renders q a Euclidean four-vector. However, k remains a Lorentz vector, and therefore the argument of $\bar{\chi}^{(P)}$ becomes complex. Furthermore, the vector $\bar{p}_2 \equiv -ip_2$ is not Euclidean, as it would be if it were in its rest frame. Since the argument of $\bar{\chi}^{(P)}$ is not q but instead q-k, one must check to see that no singularities interfere with the rotation of the contour. By referring to the diagram in section I-C showing the singularity structure in the q_o-plane, one can see that the Wick rotation is allowed provided that

$$t < 2m - M_B . \tag{I-N-13}$$

O. MAGNETIC MOMENT OF THE VECTOR MESONS

The magnetic moment of the vector mesons can be calculated from the electromagnetic current matrix elements between two vector meson states.

The first step is to express the current matrix elements in terms of form factors. Using the requirements of Lorentz invariance, parity, time reversal, hermiticity, and charge conservation, it can be shown that the current matrix elements of a spin j particle can be expressed in terms of 2j+1 form factors. For spin one, the current can be expressed as

$$\begin{aligned} \langle P_2 e_2 | j_\mu(0) | P_1 e_1 \rangle = \frac{1}{(2\pi)^3} \Big\{ \big[& F_1(Q^2) e_2^+ \cdot e_1 \\ & + F_2(Q^2) e_2^+ \cdot Q \, e_1 \cdot Q \big] P_\mu \\ & + F_3(Q^2) Q_\nu \left[e_{1\mu} e_{2\nu}^+ - e_{2\mu}^+ e_{1\nu} \right] \Big\} , \end{aligned} \quad (\text{I-0-1})$$

where

$$\begin{aligned} Q &= P_2 - P_1 \\ P &= P_2 + P_1 . \end{aligned} \quad (\text{I-0-2})$$

To extract the magnetic moment, one can form a nonrelativistic wavepacket and allow it to interact with a uniform magnetic field $\vec{B}$. Let the normalized wavepacket state $|\psi e\rangle$ be localized near some momentum $\tilde{\vec{P}}$ and near some position $\tilde{\vec{r}}$. The Hamiltonian is given by

$$H = - \int d^3 \vec{r} \; \vec{J}(\vec{r}) \cdot \vec{A}(\vec{r}) , \quad (\text{I-0-3})$$

where

$$\vec{A}(\vec{r}) = - \frac{1}{2} \vec{r} \times \vec{B} . \quad (\text{I-0-4})$$

A little rearrangement yields

$$H = -\vec{\mu}\cdot\vec{B}\,, \tag{I-0-5}$$

where

$$\vec{\mu} = \frac{1}{2}\int d^3\vec{r}\;\; \vec{r}\times\vec{J}(\vec{r}). \tag{I-0-6}$$

μ is clearly the analog of the classical magnetic moment. Now calcualte the expectation value of $\vec{\mu}$ for the wavepacket.

$$\langle\psi e|\mu_i|\psi e\rangle = \frac{1}{2}\epsilon_{ijk}\int d^3r\, r_j \int \frac{d^3P_2}{2P_{20}}\,\frac{d^3P_1}{2P_{10}} \tag{I-0-7}$$
$$\times \langle\psi e|P_2 e\rangle\langle P_2 e|j_k(0)|P_1 e\rangle e^{i(\vec{P}_1-\vec{P}_2)\cdot\vec{r}}\langle P_1 e|\psi e\rangle.$$

Now write r_j as $i\frac{\partial}{\partial P_{2j}}$ acting on the exponential, and integrate by parts. Since the wavepacket is localized near $\tilde{\vec{r}}$, one can take

$$\frac{\partial}{\partial P_{2j}}\langle\psi e|P_2 e\rangle \approx i\tilde{r}_j\langle\psi e|P_2 e\rangle. \tag{I-0-8}$$

One then has

$$\langle\psi e|\vec{\mu}|\psi e\rangle = \frac{(2\pi)^3}{2M}\left\{\frac{1}{2}\tilde{\vec{r}}\times\langle\tilde{\vec{P}}e|\vec{J}(0)|\tilde{\vec{P}}e\rangle \right. \tag{I-0-9}$$
$$\left. -\frac{i}{2}\vec{\nabla}_{P_2}\times\langle\vec{P}_2 e|\vec{J}(0)|\tilde{\vec{P}}e\rangle\Big|_{\vec{P}_2=\tilde{\vec{P}}}\right\}.$$

The first term corresponds to the orbital contribution to the magnetic moment, while the intrinsic magnetic moment is given by the second term. Using eq. (I-0-1), one finds

$$\vec{\mu}_{\text{intrinsic}} = \frac{-i}{2M} \vec{e}^{\,*} \times \vec{e}\; F_3(0). \tag{I-0-10}$$

One usually defines

$$\mu \equiv \langle J J | \mu_z | J J \rangle . \tag{I-0-11}$$

Using the polarization vector

$$\vec{e}\,(J_z = +1) = \frac{1}{\sqrt{2}}\,(1, i, 0), \tag{I-0-12}$$

one obtains

$$\mu = \frac{F_3(0)}{2M} . \tag{I-0-13}$$

To calculate $F_3(0)$, one can choose the following convenient kinematic configuration:

$$\begin{aligned} P_1 &= (0,0,0,\ iM) \\ e_1 &= (0,0,1,\ 0) \\ P_2 &= (-t,\ 0,0,\ i\sqrt{M^2 + t^2}) \\ e_2 &= \left(\sqrt{1+\frac{t^2}{M^2}}, 0, 0, \frac{-it}{M}\right). \end{aligned} \tag{I-0-14}$$

From eq. (I-0-1), it can be seen that in this configuration the current matrix element has only one non-zero component:

$$\langle P_2 e_2 | j_3(0) | P_1 e_1 \rangle = -\frac{t}{(2\pi)^3} F_3(Q^2). \tag{I-0-15}$$

So the magnetic moment μ can be calculated from

$$\mu = -\lim_{t \to 0} \frac{(2\pi)^3}{2Mt} \langle P_2 e_2 | j_3(0) | P_1 e_1 \rangle. \tag{I-0-16}$$

The current matrix element is calculated in the same way as before, except this time both B-S wavefunctions have charge conjugation number -1. So

$$\mu = -\lim_{t \to 0} \frac{1}{2Mt} \int \frac{d^4q}{(2\pi)^4} \, \mathrm{Tr} \Big\{ \overline{\chi}(p_2, q + \tfrac{1}{2}Q, e_2) \tag{I-0-17}$$

$$i(e_1 - e_2) T_3 \; \chi(p_1, q, e_1) \; i[i\gamma \cdot (q - p_1) + m] \Big\}$$

$$+ \text{ corrections.}$$

The above equation can be Wick rotated with considerably more success than the equation for $\overline{\beta}$ (eq. (I-N-12)). The vector Q is not altered by the Wick rotation, and thus the argument of $\overline{\chi}$ becomes complex. The situation is simplified by the fact that $Q \to 0$ as $t \to 0$, so a first order Taylor expansion of $\overline{\chi}$ is sufficient. In the limit as $t \to 0$, there are no singularities in the path of the Wick rotation.

P. PERTURBATIONS OF THE BOUND STATE BETHE-SALPETER EQUATION

If one defines

$$V(P) = H(P) - G(P), \tag{I-P-1}$$

then the B-S equation takes on the very simple form

$$\begin{gathered} V(P_B)\,\chi(P_B,\lambda) = 0 \\ \bar{\chi}^T(P_B,\lambda)\,V(P_B) = 0, \end{gathered} \tag{I-P-2}$$

and the normalization condition becomes

$$\bar{\chi}^T(P_B,\lambda)\,\frac{\partial V(P_B)}{\partial P_{B\mu}}\,\chi(P_B,\lambda) = 2i\,P_{B\mu}. \tag{I-P-3}$$

In this simple form it is very easy to derive the lowest order effect of perturbations in V(P), which may be due to perturbations in H or G or both.

Suppose eqs. (I-P -2) and (I-P-3) hold for some V^O, with an eigen-momentum $P_B{}^O$ and a bound state wavefunction $\chi^o(P_B^o,\lambda)$. Now consider a slightly perturbed function V given by

$$V(P) = V^o(P) + \Delta V(P). \tag{I-P-4}$$

The new solution will have a new bound state momentum P_B related by

$$P_B = P_B^o + \Delta P_B. \tag{I-P-5}$$

So to lowest order in the perturbation,

$$V(P_B) = V^\circ(P_B^\circ) + \Delta V(P_B^\circ) + \Delta P_{B\mu} \frac{\partial V^\circ(P_B^\circ)}{\partial P_{B\mu}^\circ} . \tag{I-P-6}$$

The perturbed wavefunction can be expressed as

$$\chi(P_B,\lambda) = \chi^\circ(P_B^\circ,\lambda) + \Delta\chi(P_B,\lambda) \tag{I-P-7}$$

Inserting these relations in the B-S equation and keeping only terms of first order,

$$\left[\Delta V(P_B^\circ) + \Delta P_{B\mu} \frac{\partial V^\circ(P_B^\circ)}{\partial P_{B\mu}} \right] \chi^\circ(P_B^\circ,\lambda) \tag{I-P-8}$$

$$+ V^\circ(P_B^\circ)\,\Delta\chi(P_B,\lambda) = 0 .$$

Now multiply on the left by $\overline{\chi}^\circ(P_B^\circ,\lambda)$. The final term vanishes, and the middle term is simplified by the normalization condition. The result is

$$2P_{B\mu}^\circ \,\Delta P_{B\mu} = i\,\overline{\chi}^{\circ\,T}(P_B^\circ,\lambda)\,\Delta V(P_B^\circ)\,\chi(P_B^\circ,\lambda). \tag{I-P-9}$$

Using $P_B^{\;2} = -M_B^{\;2}$, this relation can be written as

$$\Delta(M_B^2) = -i\,\overline{\chi}^{\circ\,T}(P_B^\circ,\lambda)\,\Delta V(P_B^\circ)\,\chi(P_B^\circ,\lambda). \tag{I-P-10}$$

There are two applications of this formula which will be useful later. The first application is to calculate the effect of varying one of the quark masses. We will take G(P) as being independent of the quark masses, as it is in the ladder approximation. Further, we will assume the "bare" form of the inverse propagators occuring in H(P):

$$H_{\alpha\beta\gamma\delta}(P;q,q') = (2\pi)^4\,\delta^4(q'-q)\, i\,[i\gamma\cdot(q+p)+m_1]_{\alpha\gamma} \\ i\,[i\gamma\cdot(q-p)+m_2]_{\delta\beta}\,. \qquad \text{(I-P-11)}$$

It then follows that

$$\frac{\partial M_B^2}{\partial m_1} = \int\frac{d^4q}{(2\pi)^4}\,\mathrm{Tr}\,\{\overline{\chi}^{\circ}(p,q,\lambda)\,\chi^{\circ}(p,q,\lambda) \\ \times\, i\,[i\gamma\cdot(q-p)+m_2]\} \qquad \text{(I-P-12)}$$

$$\frac{\partial M_B^2}{\partial m_2} = \int\frac{d^4q}{(2\pi)^4}\,\mathrm{Tr}\,\{\overline{\chi}^{\circ}(p,q,\lambda)\, i\,[i\gamma\cdot(q+p)+m_1] \\ \times\,\chi^{\circ}(p,q,\lambda)\}\,.$$

If $m_1 = m_2$ and charge conjugation symmetry holds, then eq. (I-E-17) can be used to show that the two expressions above are equal. The equation can be Wick rotated, and the result is

$$\frac{\partial M_B^2}{\partial m_1} = \int\frac{d^4\bar{q}}{(2\pi)^4}\,\mathrm{Tr}\,\{\overline{\chi}^{\circ}(\bar{p},\bar{q},\lambda)\,\chi^{\circ}(\bar{p},\bar{q},\lambda) \\ \times\,[\gamma\cdot(i\bar{q}+\bar{p})+m_2]\}\,. \qquad \text{(I-P-13)}$$

The second application is to calculate the effect of varying the coupling constant. Suppose

$$G(P) = g^2\,\widetilde{G}(P)\,, \qquad \text{(I-P-14)}$$

as is the case in the ladder approximation. Then if one varies g^2,

$$\Delta(M_B^2) = i\,\Delta g^2\,\bar{\chi}^{\circ T}(P_B^\circ,\lambda)\,\tilde{G}(P_B^\circ)\,\chi^\circ(P_B^\circ,\lambda). \tag{I-P-15}$$

Using the B-S equation to replace G(P) by H(P) / g^2, the equation can be rewritten

$$\frac{\partial M_B^2}{\partial g^2} = \frac{i}{g^2}\,\bar{\chi}^{\circ T}(P_B^\circ,\lambda)\,H(P_B^\circ)\,\chi^\circ(P_B^\circ,\lambda). \tag{I-P-16}$$

Again assuming the "bare" form of H(P), this expression becomes

$$\frac{\partial M_B^2}{\partial g^2} = \frac{i}{g^2}\int\frac{d^4q}{(2\pi)^4}\,\mathrm{Tr}\,\Big\{\bar{\chi}^\circ(p,q,\lambda)\,i\,[i\gamma\cdot(q+p)+m_1] \times \chi^\circ(p,q,\lambda)\,i\,[i\gamma\cdot(q-p)+m_2]\Big\}. \tag{I-P-17}$$

Wick rotating,

$$\frac{\partial M_B^2}{\partial g^2} = \frac{1}{g^2}\int\frac{d^4\bar{q}}{(2\pi)^4}\,\mathrm{Tr}\,\Big\{\bar{\chi}^\circ(\bar{p},\bar{q},\lambda)\,[\gamma\cdot(i\bar{q}-\bar{p})+m_1] \times \chi^\circ(\bar{p},\bar{q},\lambda)\,[\gamma\cdot(i\bar{q}+\bar{p})+m_2]\Big\}. \tag{I-P-18}$$

PART II:

SOLUTION OF THE BETHE-SALPETER EQUATION

A. INTRODUCTION

This part will describe the mathematical techniques which were used to obtain numerical solutions to the bound state B-S equation.

Numerical solutions to the B-S equation for deeply bound spin-½ quarks have been investigated by Narayanaswamy and Pagnamenta[13,14], and by Sundaresan and Watson[15]. Both sets of authors used an expansion of the B-S wavefunction, introduced by Gourdin[16], in terms of scalar and three-vector functions. Here the analysis is done using only Lorentz invariant functions. The method is otherwise similar to that used by Narayanaswamy and Pagnamenta.

For simplicity, we will seek solutions to the B-S equation for the bound state of a quark-antiquark pair of equal mass. Such solutions should apply to the nonstrange mesons. The λ quark is presumably only slightly heavier than the p and n quarks, so the mass difference will be treated as a first order perturbation.

Also for simplicity, our attention will be confined to the pseudoscalar and vector mesons. As will be seen, however, the vector mesons will appear with O(4) partners, or daughters, with $J^{PC} = 0^{+-}$. These problematical daughter states will also be studied.

In ladder approximation, the desired B-S equation for scalar, pseudoscalar, or neutral vector interactions is given by eqs. (I-I-10) - (I-I-12). These equations are too singular to be solved by the Fredholm method, and the numerical techniques which will be used are nothing more than approximations to the Fredholm solution. However, if one Feynman regulator is included in the propagator for the exchanged particle, then the problem will have Fredholm solutions. The Wick rotated B-S equation for a scalar interaction is then written as

$$\chi(\bar{p},\bar{q},e) = g^2\, F(\bar{p},\bar{q})\, S_-(\bar{p},\bar{q}) \times \int \frac{d^4k}{(2\pi)^4}\, \chi(\bar{p},\bar{k},e)\, \Delta(\bar{k}-\bar{q})\, S_+(\bar{p},\bar{q}), \tag{II-A-1}$$

where

$$F(\bar{p},\bar{q}) = \frac{1}{(m^2+q^2-p^2)^2 + 4(\bar{p}\cdot\bar{q})^2}, \tag{II-A-2}$$

$$S_\pm(\bar{p},\bar{q}) = \gamma\cdot(i\bar{q} \pm \bar{p}) - m, \tag{II-A-3}$$

and

$$\Delta(\bar{k}-\bar{q}) = \frac{1}{(\bar{k}-\bar{q})^2+\mu^2} - \frac{1}{(\bar{k}-\bar{q})^2+\Lambda^2} \tag{II-A-4}$$

where Λ is a cutoff mass. The symbols q and p refer to the magnitudes of the Euclidean four-vectors $\bar{q}$ and $\bar{p}$, respectively. (Hence, $p = \frac{1}{2} M_B$.) For a pseudoscalar or neutral vector interaction, the substitutions (I-I-11) and (I-I-12) still apply.

Our main goal is to explore the possibility of the existence of deeply bound solutions to the B-S equation which in some way simulate the nonrelativistic bound state problem. According to the idea of Morpurgo[3], we should imagine heavy quarks moving nonrelativistically at the bottom of a broad, deep potential well. When the B-S equation in the above form is solved, it is found that the solutions have the undesirable property that the wavefunctions do not fall off until $q \approx m$. (The solution can be expected to have this property on the grounds that, when $\bar{q}$ is neglected compared to m in $F(\bar{p},\bar{q})$ and $S_{\pm}(\bar{p},\bar{q})$, the equation becomes more singular then the Fredholm solution allows.) Thus, in order to explore Morpurgo's conjecture, one must assume that the interaction between quarks is smoother (in coordinate space) than that given by eq. (II-A-4).

If one more regulator is added to $\Delta(\bar{k}-\bar{q})$, so that

$$\Delta(\bar{k}) \longrightarrow \frac{1}{k^6} \quad \text{as} \quad k \longrightarrow \infty, \tag{II-A-5}$$

then solutions can be found which fall off at values of q much less than m. Without introducing any new parameters, this regulator can take the form

$$\Delta(\bar{k}-\bar{q}) = \frac{1}{(\bar{k}-\bar{q})^2+\mu^2} - \frac{1}{(\bar{k}-\bar{q})^2+\Lambda^2} - \frac{\Lambda^2-\mu^2}{[(\bar{k}-\bar{q})^2+\Lambda^2]^2} . \tag{II-A-6}$$

This form is not unique, but it has the desired properties.

Solutions have been obtained using both the singly and the doubly regulated form of the exchange propagator.

B. REDUCTION OF THE EQUATION

To solve the B-S equation (II-A-1), the first step is to expand $\chi(p,q,e)$ in Lorentz invariant functions, as described in section I-F:

$$\chi_{\alpha\beta}(\bar{p},\bar{q},e) = \sum_i \chi^{(i)}(\bar{q}^2, \bar{p}\cdot\bar{q})\, M^{(i)}_{\alpha\beta}(\bar{p},\bar{q},e). \tag{II-B-1}$$

The values of $M^{(i)}(p,q,e)$ come from section I-F, and are tabulated in Table II-B-1, at the end of this section. The B-S equation for a scalar interaction then becomes

$$\begin{aligned}\sum_i \chi^{(i)}(\bar{q}^2, \bar{p}\cdot\bar{q})\, M^{(i)}(\bar{p},\bar{q},e) \quad &= \quad g^2\, F(\bar{p},\bar{q}) \\ &\times S_-(\bar{p},\bar{q}) \int \frac{d^4\bar{k}}{(2\pi)^4} \sum_j \chi^{(j)}(\bar{k}^2, \bar{k}\cdot\bar{p}) \\ &\times M^{(j)}(\bar{p},\bar{k},e)\, \Delta(\bar{k}-\bar{q})\, S_+(\bar{p},\bar{q}).\end{aligned} \tag{II-B-2}$$

For a pseudoscalar interaction, the equation is modified by

$$M^{(j)}(\bar{p},\bar{k},e) \longrightarrow -\gamma_5\, M^{(j)}(\bar{p},\bar{k},e)\,\gamma_5 \,. \tag{II-B-3}$$

By examining the $M^{(j)}$ listed in Table II-B-1, one sees that the expression on the right is just a constant times $M^{(j)}(\bar{p},\bar{k},e)$. So for a pseudoscalar interaction, the B-S equation (II-B-2) is modified by

$$M^{(j)}(\bar{p},\bar{k},e) \longrightarrow f_j^{(P)}\, M^{(j)}(\bar{p},\bar{k},e). \tag{II-B-4}$$

The values of $f_j^{(P)}$ are listed in Table II-B-2. For a vector interaction, the substitution is

$$M^{(j)}(\bar{p},\bar{k},e) \longrightarrow -\gamma_\mu M^{(j)}(\bar{p},\bar{k},e)\gamma_\mu$$
$$= f_j^{(V)} M^{(j)}(\bar{p},\bar{k},e). \tag{II-B-5}$$

The values of $f_j^{(V)}$ are contained in the same table. If one defines

$$f_j^{(S)} \equiv 1, \tag{II-B-6}$$

for a scalar interaction, then eq. (II-B-2) can be generalized to all three interactions by inserting the factor f_j in front of the factor $M^{(j)}(\bar{p},\bar{k},e)$.

The next step is to isolate a single term on the left hand side, which may be done by finding matrices $\tilde{M}^{(i)}(\bar{p},\bar{q},e)$ which have the property that

$$\mathrm{Tr}\left[\tilde{M}^{(i)}(\bar{p},\bar{q},e)\, M^{(j)}(\bar{p},\bar{q},e)\right] = \delta_{ij}. \tag{II-B-7}$$

One can express the $\tilde{M}^{(i)}$ as linear combinations of the $M^{(i)}$:

$$\tilde{M}^{(i)}(\bar{p},\bar{q},e) = \sum_j C_{ij}(\bar{p},\bar{q},e)\, M^{(j)}(\bar{p},\bar{q},e). \tag{II-B-8}$$

It then follows that

$$\sum_k C_{ik}(\bar{p},\bar{q},e)\,\mathrm{Tr}\left[M^{(k)}(\bar{p},\bar{q},e)\, M^{(j)}(\bar{p},\bar{q},e)\right] \tag{II-B-9}$$
$$= \delta_{ij}.$$

So the task is finished by defining

$$B_{ij}(\bar{p},\bar{q},e) = \mathrm{Tr}\left[M^{(i)}(\bar{p},\bar{q},e)\, M^{(j)}(\bar{p},\bar{q},e)\right], \tag{II-B-10}$$

and then

$$C(\bar{p},\bar{q},e) = B^{-1}(\bar{p},\bar{q},e), \tag{II-B-11}$$

where the inverse refers to matrix inversion. The values of the functions $\tilde{M}^{(i)}(\bar{p},\bar{q},e)$ are listed in Table II-B-3.

The reduction of eq. (II-B-2) is then accomplished by multiplying both sides by $M^{(i')}(\bar{p},\bar{q},e)$ and then taking the trace. The result is

$$\begin{aligned} \chi^{(i)}(\bar{q}^2, \bar{p}\cdot\bar{q}) &= g^2\, F(\bar{p},\bar{q}) \sum_j \int \frac{d^4\bar{k}}{(2\pi)^4} \\ &\times f_j\, H^{ij}(\bar{p},\bar{q},\bar{k},e)\, \chi^{(j)}(\bar{k}^2, \bar{p}\cdot\bar{k})\, \Delta(\bar{k}-\bar{q}), \end{aligned} \tag{II-B-12}$$

where

$$\begin{aligned} H^{ij}(\bar{p},\bar{q},\bar{k},e) &= \mathrm{Tr}\left[\tilde{M}^{(i)}(\bar{p},\bar{q},e)\, S_-(\bar{p},\bar{q})\right. \\ &\times \left. M^{(j)}(\bar{p},\bar{k},e)\, S_+(\bar{p},\bar{q})\right]. \end{aligned} \tag{II-B-13}$$

The B-S equation is rotationally invariant, so at some point the dependence on the polarization tensor e should disappear. However, the dependence on e does not disappear in the above equation when the trace is taken. The reason is that the same functions H^{ij} would appear if eq. (II-A-1) were solved with an interation function $\Delta(k-q)$ which violated rotational invariance. Thus the functions H^{ij} contain more information then we will finally need, so they will not be tabulated.

The Lorentz invariant functions $\chi^{(i)}$ depend on the two variables $\bar{q}^2$ and $\bar{p}\cdot\bar{q}$. Therefore two of the four integrations in eq. (II-B-12) are trivial, so it can be reduced to a system of coupled two dimensional integral equations. However, for numerical solutions, it is highly advantageous to have one dimensional integral equations. This reduction can be accomplished by exploiting the approximate O(4) symmetry which exists for the deeply bound states.

To understand the meaning of the O(4) limit, consider the B-S equation, in the form of eq. (I-I-10) for example, in the rest frame of the bound state. Then

$$\bar{p} = (\vec{0}, \tfrac{1}{2}M_B), \qquad \text{(II-B-14)}$$

and so $\bar{p} \rightarrow 0$ as $M_B \rightarrow 0$. When $M_B = 0$, this equation possesses exact O(4) symmetry, which means simply that if $\chi(\bar{p} = 0, \bar{q})$ satisfies the integral equation, then

$$\chi'(\bar{p}=0,\bar{q}) \equiv \chi(\bar{p}=0, R\bar{q}),$$

where R represents any four dimensional rotation, also satisfies the equation. It follows that the solutions correspond to O(4) representations, and can be expressed in terms of O(4) eigenfunctions. Note that the O(4) limit does not necessarily describe actual zero mass bound states, since zero mass bound states cannot be treated in their rest frame. Rather, the O(4) limit is a mathematical limit of the equation for massive bound states. Provided that the limit is approached smoothly, states of very small mass (compared to the masses of the constituents) can be described accurately by a small number of terms of an expansion in O(4) eigenfunctions.

Confining our attention to the rest frame of the bound state, $\chi^{(i)}(\bar{q}^2, \bar{p}\cdot\bar{q})$ depends only on q and β_q, the angle between $\bar{q}$ and the 4-axis. It can therefore be expanded

$$\chi^{(i)}(\bar{q}^2, \bar{p}\cdot\bar{q}) = \sum_{n=0}^{\infty} \chi_n^{(i)}(q)\, \mathcal{C}^0_n(\beta_q), \qquad \text{(II-B-15)}$$

where $\mathcal{C}^0_n(\beta)$ is defined and discussed in Appendix B. Using the orthornormality relation (A-B-16), the B-S equation (II-B-13) can be reduced to

$$\chi_n^{(i)}(q) = g^2 \sum_j \sum_{n'=0}^{\infty} \int_0^\infty dk\, K^{ij}_{nn'}(q,k)\, \chi^{(j)}_{n'}(k), \qquad \text{(II-B-16)}$$

where

$$K^{ij}_{nn'}(q,k) = \frac{k^3}{(2\pi)^4} \int_0^\pi \sin^2\beta_q\, d\beta_q\, \mathcal{C}^0_n(\beta_q)\, F(\bar{p},\bar{q}) \times \int d\Omega_k\, \mathcal{C}^0_{n'}(\beta_k)\, \Delta(\bar{k}-\bar{q})\, H^{ij}(\bar{p},\bar{q},\bar{k},e). \qquad \text{(II-B-17)}$$

With this integration, the dependence on the polarization tensor e drops out. The $K^{ij}_{nn'}(q,k)$ will be evaluated in the following section.

TABLE II-B-1

VALUES OF $M^{(i)}(\overline{p},\overline{q},e)$

Pseudoscalar Bound States:

$$M^{(1)} = \gamma_5$$
$$M^{(2)} = \gamma\cdot\overline{p}\ \gamma_5$$
$$M^{(3)} = \gamma\cdot\overline{q}\ \gamma_5$$
$$M^{(4)} = \varepsilon_{\mu\nu\lambda\sigma}\ \overline{p}_\mu \overline{q}_\nu\ \sigma_{\lambda\sigma}$$

Scalar Bound States:

$$M^{(1)} = i$$
$$M^{(2)} = \gamma\cdot\overline{p}$$
$$M^{(3)} = \gamma\cdot\overline{q}$$
$$M^{(4)} = i\ \overline{p}_\mu \overline{q}_\nu\ \sigma_{\mu\nu}$$

Vector Bound States:

$$M^{(1)} = i\overline{q}\cdot e$$
$$M^{(2)} = \overline{q}\cdot e\ \gamma\cdot\overline{p}$$
$$M^{(3)} = \overline{q}\cdot e\ \gamma\cdot\overline{q}$$
$$M^{(4)} = i\ \overline{q}\cdot e\ \overline{p}_\mu\ \overline{q}_\nu\ \sigma_{\mu\nu}$$
$$M^{(5)} = \gamma\cdot e$$
$$M^{(6)} = i\varepsilon_{\mu\nu\lambda\sigma}\ e_\mu\ \overline{p}_\nu\ \overline{q}_\lambda\ \gamma_\sigma\gamma_5$$
$$M^{(7)} = ie_\mu\ \overline{p}_\nu\ \sigma_{\mu\nu}$$
$$M^{(8)} = ie_\mu\ \overline{q}_\nu\ \sigma_{\mu\nu}$$

TABLE II-B-2

VALUES OF $f_j^{(P)}$ AND $f_j^{(V)}$

	j	$f_j^{(P)}$	$f_j^{(V)}$
Pseudoscalar Bound States:	1	-1	4
	2	1	-2
	3	1	-2
	4	-1	0
Scalar Bound States:	1	-1	-4
	2	1	2
	3	1	2
	4	-1	0
Vector Bound States:	1	-1	-4
	2	1	2
	3	1	2
	4	-1	0
	5	1	2
	6	1	-2
	7	-1	0
	8	-1	0

TABLE II-B-3

VALUES OF $\tilde{M}^{(i)}$ $(\bar{p},\bar{q},e)$

Pseudoscalar Bound States:

$$\tilde{M}^{(1)} = \frac{1}{4}\gamma_5$$
$$\tilde{M}^{(2)} = \frac{1}{4} D \left[\bar{p}\cdot\bar{q}\ \gamma\cdot\bar{q} - \bar{q}^2\ \gamma\cdot\bar{p}\right] \gamma_5$$
$$\tilde{M}^{(3)} = \frac{1}{4} D \left[\bar{p}\cdot\bar{q}\ \gamma\cdot\bar{p} - \bar{p}^2\ \gamma\cdot\bar{q}\right] \gamma_5$$
$$M^{(4)} = \frac{1}{16} D\ \varepsilon_{\mu\nu\lambda\sigma}\ \bar{p}_\mu\ \bar{q}_\nu\ \sigma_{\lambda\sigma},$$

where

$$D = \frac{1}{\bar{p}^2\bar{q}^2 - (\bar{p}\cdot\bar{q})^2} .$$

Scalar Bound States:

$$\tilde{M}^{(1)} = \frac{-i}{4}$$
$$\tilde{M}^{(2)} = \frac{1}{4} D \left[\bar{q}^2\ \gamma\cdot\bar{p} - \bar{p}\cdot\bar{q}\ \gamma\cdot\bar{q}\right]$$
$$\tilde{M}^{(3)} = \frac{1}{4} D \left[\bar{p}^2\ \gamma\cdot\bar{q} - \bar{p}\cdot\bar{q}\ \gamma\cdot\bar{p}\right]$$
$$\tilde{M}^{(4)} = -\frac{i}{4} D\ \bar{p}_\mu\bar{q}_\nu\ \sigma_{\mu\nu}.$$

Vector Bound States:

$$\tilde{M}^{(1)} = \frac{-i}{4\bar{q}\cdot e}$$

$$\tilde{M}^{(2)} = \frac{1}{4\bar{p}^2\bar{q}\cdot e}\ \{\gamma\cdot\bar{p} + \bar{p}\cdot\bar{q}\ D' \left[\bar{p}^2\bar{q}\cdot e\ \gamma\cdot e + \bar{p}\cdot\bar{q}\ \gamma\cdot\bar{p} - \bar{p}^2\gamma\cdot\bar{q}\right]\}$$

$$\tilde{M}^{(3)} = \frac{1}{4\bar{q}\cdot e} D' \left[\bar{p}^2\gamma\cdot\bar{q} - \bar{p}\cdot\bar{q}\ \gamma\cdot\bar{p} - \bar{p}^2\bar{q}\cdot e\gamma\cdot e\right]$$

$$\tilde{M}^{(4)} = -\frac{i}{4} D' \left[e_\mu \bar{p}_\nu \sigma_{\mu\nu} + \frac{1}{\bar{q}\cdot e} \bar{p}_\mu \bar{q}_\nu \sigma_{\mu\nu} \right]$$

$$\tilde{M}^{(5)} = \frac{1}{4} \gamma\cdot e + \frac{1}{4} D' \left[\bar{p}^2 (\bar{q}\cdot e)^2 \gamma\cdot e + \bar{p}\cdot\bar{q}\ \bar{q}\cdot e\ \gamma\cdot\bar{p} - \bar{p}^2 \bar{q}\cdot e\ \gamma\cdot\bar{q} \right]$$

$$\tilde{M}^{(6)} = -\frac{i}{4} D' \varepsilon_{\mu\nu\lambda\sigma} e_\mu \bar{p}_\nu \bar{q}_\lambda \gamma_\sigma \gamma_5$$

$$\tilde{M}^{(7)} = -\frac{i}{4} D' \left[\bar{q}^2 e_\mu \bar{p}_\nu \sigma_{\mu\nu} - \bar{p}\cdot\bar{q}\ e_\mu \bar{q}_\nu \sigma_{\mu\nu} + \bar{q}\cdot e\ \bar{p}_\mu \bar{q}_\nu \sigma_{\mu\nu} \right]$$

$$\tilde{M}^{(8)} = \frac{i}{4} D' \left[\bar{p}\cdot\bar{q}\ e_\mu \bar{p}_\nu \sigma_{\mu\nu} - \bar{p}^2 e_\mu \bar{q}_\nu \sigma_{\mu\nu} \right] ,$$

where

$$D' = \frac{1}{\bar{p}^2 \bar{q}^2 - (\bar{p}\cdot\bar{q})^2 - \bar{p}^2 (\bar{q}\cdot e)^2} .$$

C. EVALUATION OF $K^{ij}_{nn'}(q,k)$

The quantity $K^{ij}_{nn'}(q,k)$ has been defined by eq. (II-B-17) in the previous section. This section will show how to carry out the integrations to obtain explicit expressions. We will rely heavily on the properties of O(4) eigenfunctions described in Appendix B.

1) INTEGRATION OVER $d\Omega_k$

The first step is to integrate over $d\Omega_k$. Define

$$I^{ij}_{n'}(q,k,\beta_q) \equiv \int d\Omega_k\, C^{0}_{n'}(\beta_k)\, \Delta(\bar{k}-\bar{q})\, H^{ij}(\bar{p},\bar{q},\bar{k},e), \tag{II-C-1}$$

where

$$\bar{P} = (0,0,0,p) \tag{II-C-2}$$

with

$$p = \tfrac{1}{2} M_B . \tag{II-C-3}$$

Depending on j, $H^{ij}(\bar{p},\bar{q},\bar{k},e)$ may be independent of $\bar{k}_\mu$, linear in $\bar{k}_\mu$, or linear in $\bar{k}_\mu \bar{k}_\nu$. These cases will be treated one at a time.

If $H^{ij}(\bar{p},\bar{q},\bar{k},e)$ is independent of $\bar{k}_\mu$, then we can set

$$\tilde{H}^{ij}_{1}(q,\beta_q) \equiv H^{ij}(\bar{p},\bar{q},\bar{k},e), \tag{II-C-4}$$

and take the factor outside the integral. The independence of e will be discussed later. Then define

$$I^{1}_{n'}(q,k,\beta_q) \equiv \int d\Omega_k\, C^{0}_{n'}(\beta_k)\, \Delta(\bar{k}-\bar{q}). \tag{II-C-5}$$

This integral can be carried out by expanding the propagator $\Delta(\bar{k}-\bar{q})$ in four dimensional spherical harmonics. Let $\tilde{\Delta}(\bar{k},\mu)$ be the simple Feynman propagator

$$\tilde{\Delta}(\bar{k},\mu) \equiv \frac{1}{\bar{k}^2+\mu^2} . \qquad \text{(II-C-6)}$$

The expansion for this propagator is given in Appendix B as eq. (A-B-17), and it can be written as

$$\tilde{\Delta}(\bar{k}-\bar{q},\mu) = 8\pi^2 \sum_{n\ell m} \tilde{\Delta}_n(q,k,\mu) \times Y^*_{n\ell m}(\hat{k})\, Y_{n\ell m}(\hat{q}), \qquad \text{(II-C-7)}$$

where

$$\tilde{\Delta}_n(q,k,\mu) = \frac{1}{n+1}\,\frac{(r-s)^n}{(r+s)^{n+2}} , \qquad \text{(II-C-8)}$$

with

$$r = \sqrt{(k+q)^2+\mu^2}$$
$$s = \sqrt{(k-q)^2+\mu^2} . \qquad \text{(II-C-9)}$$

The doubly regularized propagator defined by eq. (II-A-6) can be written as

$$\Delta(\bar{k}) = \tilde{\Delta}(\bar{k},\mu) - \tilde{\Delta}(\bar{k},\Lambda) + (\Lambda^2-\mu^2)\frac{\partial}{\partial\Lambda^2}\tilde{\Delta}(\bar{k},\Lambda) . \qquad \text{(II-C-10)}$$

This propagator can then be expanded as

$$\Delta(\bar{k}-\bar{q}) = 8\pi^2 \sum_{n\ell m} \Delta_n(q,k) \times Y^*_{n\ell m}(\hat{k})\, Y_{n\ell m}(\hat{q}), \qquad \text{(II-C-11)}$$

with

$$\Delta_n(q,k) = \tilde{\Delta}_n(q,k,\mu) - \tilde{\Delta}_n(q,k,\Lambda)$$

$$+ (\Lambda^2-\mu^2)\frac{\partial}{\partial\Lambda^2}\tilde{\Delta}_n(q,k,\Lambda)$$

$$= \frac{1}{n+1}\left[\frac{(r-s)^n}{(r+s)^{n+2}} - \frac{(r'-s')^n}{(r'+s')^{n+2}} \right. \tag{II-C-12}$$

$$\left. - \frac{\Lambda^2-\mu^2}{r's'}\frac{(r'-s')^n}{(r'+s')^{n+2}}\right],$$

with

$$r' = \sqrt{(k+q)^2 + \Lambda^2}$$

$$s' = \sqrt{(k-q)^2 + \Lambda^2} \quad . \tag{II-C-13}$$

For the singly regulated propagator, one merely omits the last term of eq. (II-C-12). The integration is now trivial, requiring only the orthonormality relations for the four dimensional spherical harmonics. The result is

$$I^1_{n'}(q,k,\beta_q) = 8\pi^2\,\Delta_{n'}(q,k)\,\mathcal{C}^0_{n'}(\beta_q)$$

$$= 4(2\pi)^{3/2}\,\Delta_{n'}(q,k)\,C^1_{n'}(\cos\beta_q), \tag{II-C-14}$$

where C represents a Gegenbauer polynomial, defined in Appendix B. Then for this case,

$$I^{ij}_{n'}(q,k,\beta_q) = I^1_{n'}(q,k,\beta_q)\,\tilde{H}^{ij}_1(q,\beta_q). \tag{II-C-15}$$

Next consider the case when $H^{ij}(p,q,k,e)$ is linear in $\bar{k}_\mu$. It is then necessary to integrate

$$I^2_{n',\mu}(\bar{q},k) \equiv \int d\Omega_k\, \mathcal{C}^0_{n'}(\beta_k)\, \Delta(\bar{k}-\bar{q})\, \bar{k}_\mu . \tag{II-C-16}$$

First consider the more general integral

$$T^n_{\mu_1\mu_2\ldots\mu_n}(\bar{q},k) \equiv \int d\Omega_k\, \Delta(\bar{k}-\bar{q})\, \{\hat{k}_{\mu_1}\ldots\hat{k}_{\mu_n}\}, \tag{II-C-17}$$

where the curly brackets indicate the traceless symmetric tensor formed from $\hat{k}_\mu$ and $\delta_{\mu\nu}$, with the leading term $\hat{k}_{\mu_1}\ldots\hat{k}_{\mu_n}$ (see Appendix B). $T_{\mu_1\ldots\mu_n}$ is a traceless symmetric tensor constructed from the four-vector $\bar{q}_\mu$, so

$$T^n_{\mu_1\ldots\mu_n}(\bar{q},k) = F_n(q,k)\, \{\hat{q}_{\mu_1}\ldots\hat{q}_{\mu_n}\}, \tag{II-C-18}$$

where $F_n(q,k)$ is some function of q and k (the magnitudes of the four-vectors) only. To find $F_n(q,k)$, one can choose the simple case of $\mu_1 = \mu_2 = \ldots = \mu_n = 4$. Then use formula (A-B-29) in Appendix B to show that this integral is proportional to the one already done (eq. (II-C-14)), so $F_n(q,k)$ is determined.

$$T^n_{\mu_1\ldots\mu_n}(\bar{q},k) = 8\pi^2\, \Delta_n(q,k)\, \{\hat{q}_{\mu_1}\ldots\hat{q}_{\mu_n}\}. \tag{II-C-19}$$

The next step is to convert $I^2_{n',\mu}(q,k)$ to the general form $T^n_{\mu_1\ldots\mu_n}(\bar{q},k)$. Use eq. (A-B-29) to express $\mathcal{C}^0_{n'}(\beta_k)$ in terms of $\{\hat{k}_4\ldots\hat{k}_4\}^{(n)}$ (the superscript n indicates the rank of the tensor), and then use eq. (A-B-24) to multiply this tensor by $\hat{k}_\mu$. The integration can then be carried out using the general form $T^n_{\mu_1\ldots\mu_n}(\bar{q},k)$, and the answer involves terms of the form $\{\hat{q}_4\ldots\hat{q}_4\}$ and $\{\hat{q}_\mu\hat{q}_4\ldots\hat{q}_4\}$. Use eqs. (A-B-29) and (-36) to express these quantities in terms of Gegenbauer polynomials. The result is then

$$I^{2}_{n'\mu}(\bar{q},k) = I^{2,1}_{n'}(q,k,\beta_q)\,\bar{q}_\mu + I^{2,2}_{n'}(q,k,\beta_q)\,\bar{p}_\mu\,, \qquad \text{(II-C-20)}$$

where

$$I^{2,1}_{n'}(q,k,\beta_q) = \frac{4(2\pi)^{3/2}}{n'+1}\,\frac{k}{q}\Big[\Delta_{n'+1}(q,k) \times C^2_{n'}(\cos\beta_q) - \Delta_{n'-1}(q,k)\,C^2_{n'-2}(\cos\beta_q)\Big]\,, \qquad \text{(II-C-21)}$$

and

$$I^{2,2}_{n'}(q,k,\beta_q) = \frac{4(2\pi)^{3/2}}{n'+1}\,\frac{k}{p}\Big[\Delta_{n'-1}(q,k) - \Delta_{n'+1}(q,k)\Big]C^2_{n'-1}(\cos\beta_q)\,. \qquad \text{(II-C-22)}$$

To finish, define

$$\begin{aligned} H^{ij}_1(q,\beta_q) &\equiv H^{ij}(\bar{p},\bar{q},\bar{q},e) \\ H^{ij}_2(q,\beta_q) &\equiv H^{ij}(\bar{p},\bar{q},\bar{p},e)\,. \end{aligned} \qquad \text{(II-C-23)}$$

Then

$$I^{ij}_{n'}(q,k,\beta_q) = \sum_{s=1}^{2} I^{2,s}_{n'}(q,k,\beta_q)\,\tilde{H}^{ij}_s(q,\beta_q)\,. \qquad \text{(II-C-24)}$$

Finally, consider the case when $H^{ij}(p,q,k,e)$ is linear in $\bar{k}_\mu\bar{k}_\nu$. It is then necessary to integrate

$$I^{3}_{n'\mu\nu}(\bar{q},k) \equiv \int d\Omega_k\, C^0_{n'}(\beta_k)\,\Delta(\bar{k}-\bar{q})\,\bar{k}_\mu\bar{k}_\nu\,. \qquad \text{(II-C-25)}$$

This integral is carried out by the same techniques as the previous one. The result is

$$I^{3}_{n'\mu\nu}(\bar{q},k) = I^{3,1}_{n'}(q,k,\beta_q)\,\bar{q}_\mu\bar{q}_\nu$$
$$+ I^{3,2}_{n'}(q,k,\beta_q)\,\delta_{\mu\nu} + I^{3,3}_{n'}(q,k,\beta_q) \qquad \text{(II-C-26)}$$
$$\times(\bar{p}_\mu\bar{q}_\nu + \bar{q}_\mu\bar{p}_\nu) + I^{3,4}_{n'}(q,k,\beta_q)\,\bar{p}_\mu\bar{p}_\nu\,,$$

where

$$I^{3,1}_{n'}(q,k,\beta_q) = 8(2\pi)^{3/2}\frac{k^2}{q^2}\left[\frac{\Delta_{n'+2}(q,k)\,C^3_{n'}(\cos\beta_q)}{(n'+1)(n'+2)}\right. \qquad \text{(II-C-27)}$$
$$\left. - \frac{2\Delta_{n'}(q,k)\,C^3_{n'-2}(\cos\beta_q)}{n'(n'+2)} + \frac{\Delta_{n'-2}(q,k)\,C^3_{n'-4}(\cos\beta_q)}{n'(n'+1)}\right],$$

$$I^{3,2}_{n'}(q,k,\beta_q) = 2(2\pi)^{3/2}k^2\left[-\Delta_{n'+2}(q,k)\frac{C^2_{n'}(\cos\beta_q)}{(n'+1)(n'+2)}\right. \qquad \text{(II-C-28)}$$
$$+ \Delta_{n'}(q,k)\,\frac{n'C^1_{n'}(\cos\beta_q) + 2C^2_{n'-2}(\cos\beta_q)}{n'(n'+2)}$$
$$\left. - \Delta_{n'-2}(q,k)\,\frac{C^2_{n'-2}(\cos\beta_q)}{n'(n'+1)}\right],$$

and
$$I^{2,3}_{n'}(q,k,\beta_q) = 4(2\pi)^{3/2}\frac{k^2}{qp}\left[-\Delta_{n'+2}(q,k)\,\frac{2C^3_{n'-1}(\cos\beta_q)}{(n'+1)(n'+2)}\right. \qquad \text{(II-C-29)}$$
$$+ \Delta_{n'}(q,k)\,\frac{4C^3_{n'-3}(\cos\beta_q) + n'C^2_{n'-1}(\cos\beta_q)}{n'(n'+2)}$$

$$- \Delta_{n'-2}(q,k) \; \frac{2C^3_{n'-5}(\cos\beta_q) + (n'-1)C^2_{n'-3}(\cos\beta_q)}{n'\;(n'+1)} \Bigg] .$$

$I^{3,4}_{n'}(q,k,\beta_q)$ will not be necessary. Now let

$$H^{ij}(\bar{p},\bar{q},\bar{k},e) = A^{ij}_{\mu\nu}\,\bar{k}_\mu \bar{k}_\nu , \qquad \text{(II-C-30)}$$

and then define

$$\begin{aligned} \tilde{H}^{ij}_1(q,\beta_q) &\equiv A^{ij}_{\mu\nu}\,\bar{q}_\mu \bar{q}_\nu = H^{ij}(\bar{p},\bar{q},\bar{q},e) \\ \tilde{H}^{ij}_2(q,\beta_q) &\equiv A^{ij}_{\mu\nu}\,\delta_{\mu\nu} \\ \tilde{H}^{ij}_3(q,\beta_q) &\equiv A^{ij}_{\mu\nu}\,(\bar{p}_\mu \bar{q}_\nu + \bar{q}_\mu \bar{p}_\nu) \\ \tilde{H}^{ij}_4(q,\beta_q) &\equiv A^{ij}_{\mu\nu}\,\bar{p}_\mu \bar{p}_\nu . \end{aligned} \qquad \text{(II-C-31)}$$

As will be shown in the next section, $\tilde{H}^{ij}_4 = 0$. So

$$I^{ij}_{n'}(q,k,\beta_q) = \sum_{s=1}^{3} I^{3,s}_{n'}(q,k,\beta_q)\,\tilde{H}^{ij}_s(q,\beta_q) . \qquad \text{(II-C-32)}$$

All of the expressions for the $I^{m,s}_{n'}(q,k,\beta_q)$ are to be interpreted using the convention that $C^{\alpha}_{n}(x) = 0$ if $n < 0$. Those values of $I^{m,s}_{n'}$ which will be used are listed in Table II-C-1, at the end of this section.

2. EVALUATION OF $H^{ij}_s(q,\beta_q)$

First consider the cases of $s \neq 1$. By their definition, these values are calculated by finding $H^{ij}(\bar{p},\bar{q},\bar{k},e)$, and then carrying out one of the following substitutions:

$$\begin{aligned}
\bar{k}_\mu &\longrightarrow \bar{p}_\mu \\
\bar{k}_\mu \bar{k}_\nu &\longrightarrow \delta_{\mu\nu} \\
\bar{k}_\mu \bar{k}_\nu &\longrightarrow \bar{p}_\mu \bar{q}_\nu + \bar{q}_\mu \bar{p}_\nu \\
\bar{k}_\mu \bar{k}_\nu &\longrightarrow \bar{p}_\mu \bar{p}_\nu .
\end{aligned}$$

These operations commute, so the substitutions can be carried out on the $M^{(j)}(\bar{p},\bar{k},e)$ before the trace in eq. (II-B-13) is performed. When this is done, it is found that $M^{(j)}(p,k,e)$ undergoes a transformation

$$\begin{aligned}
M^{(j)}(\bar{p},\bar{k},e) &\longrightarrow \pm M^{(j')}(\bar{p},\bar{q},e) \\
\text{or} \quad &\longrightarrow 0 .
\end{aligned}$$

So the $s \neq 1$ values can all be expressed in terms of the $s = 1$ values. The relations are given in Table II-C-2.

For $s = 1$, one must calculate

$$\begin{aligned}
\tilde{H}_1^{ij}(q,\beta_q) = \mathrm{Tr}\big[&\tilde{M}^{(i)}(\bar{p},\bar{q},e)\, S_-(\bar{p},\bar{q}) \\
&\times M^{(j)}(\bar{p},\bar{q},e)\, S_+(\bar{p},\bar{q})\big] .
\end{aligned} \tag{II-C-33}$$

One way to proceed is to calculate the traces directly, although it is very tedious to do this by hand for the 64 cases necessary for the vector bound states. The calculation of the traces is, however, very straightforward, and a computer program was written which performs this kind of calculation in closed form.

The calculation can also be done by hand by first calculating the following relations:

$$S_-(\bar{p},\bar{q})\, S_+(\bar{p},\bar{q}) = (m^2 - q^2 - p^2) - 2im\,\gamma\cdot\bar{q} - 2\bar{p}_\mu \bar{q}_\nu \sigma_{\mu\nu}\,,$$

$$S_-(\bar{p},\bar{q})\,\gamma_5\, S_+(\bar{p},\bar{q}) = (m^2 + q^2 + p^2)\gamma_5 + 2m\,\gamma\cdot\bar{p}\,\gamma_5 - \epsilon_{\mu\nu\lambda\sigma}\bar{p}_\mu \bar{q}_\nu \sigma_{\lambda\sigma}\,,$$

$$\begin{aligned} S_-(\bar{p},\bar{q})\,\gamma_\mu\, S_+(\bar{p},\bar{q}) = {} & -2im\,\bar{q}_\mu - 2\bar{q}_\mu\,\gamma\cdot\bar{q} \\ & - 2\bar{p}_\mu\,\gamma\cdot\bar{p} + (m^2 + q^2 + p^2)\gamma_\mu \\ & - 2i\epsilon_{\mu\nu\lambda\sigma}\bar{p}_\nu \bar{q}_\lambda \gamma_\sigma \gamma_5 + 2im\,\bar{p}_\nu \sigma_{\mu\nu}\,, \end{aligned} \tag{II-C-34}$$

$$\begin{aligned} S_-(\bar{p},\bar{q})\,\gamma_\mu\gamma_5\, S_+(\bar{p},\bar{q}) = {} & 2m\,\bar{p}_\mu\gamma_5 \\ & + 2i\epsilon_{\mu\nu\lambda\sigma}\bar{p}_\nu \bar{q}_\lambda \gamma_\sigma + 2\bar{q}_\mu\,\gamma\cdot q\,\gamma_5 \\ & + 2\bar{p}_\mu\,\gamma\cdot\bar{p}\,\gamma_5 + (m^2 - q^2 - p^2)\gamma_\mu\gamma_5 \\ & - m\,\epsilon_{\mu\nu\lambda\sigma}\bar{q}_\nu \sigma_{\lambda\sigma}\,, \end{aligned}$$

$$\begin{aligned} S_-(\bar{p},\bar{q})\,\sigma_{\mu\nu}\, S_+(\bar{p},\bar{q}) = {} & 2(\bar{p}_\mu \bar{q}_\nu - \bar{q}_\mu \bar{p}_\nu) \\ & + 2\epsilon_{\mu\nu\lambda\sigma}\bar{p}_\lambda \bar{q}_\sigma \gamma_5 + 2im(\bar{p}_\mu\gamma_\nu - \bar{p}_\nu\gamma_\mu) \\ & - 2m\,\epsilon_{\mu\nu\lambda\sigma}\bar{q}_\lambda \gamma_\sigma\gamma_5 \\ & + 2\left[(\bar{q}_\mu\bar{q}_\lambda + \bar{p}_\mu\bar{p}_\lambda)\sigma_{\lambda\nu} - (\bar{q}_\nu\bar{q}_\lambda + \bar{p}_\nu\bar{p}_\lambda)\sigma_{\lambda\mu}\right] \\ & + (m^2 - q^2 - p^2)\sigma_{\mu\nu}\,. \end{aligned}$$

Using these relations, it is not difficult to calculate $S_-(\bar{p},\bar{q})\, M^j(\bar{p},\bar{q},e)\, S_+(\bar{p},\bar{q})$. This expression has the same tensor and parity properties as

$\chi(p,q,e)$, so it can be expanded in terms of the same tensor functions. Remembering eq. (II-B-7), it can be seen that the expansion coefficients are just the $H_1^{ij}(q,\beta_q)$ for which we are looking.

$$S_-(\bar{p},\bar{q})\, M^{(j)}(\bar{p},\bar{q},e)\, S_+(\bar{p},\bar{q}) = \sum_i \tilde{H}_1^{ij}(q,\beta_q)\, M^{(i)}(\bar{p},\bar{q},e). \tag{II-C-35}$$

Since the $M^{(i)}(\bar{p},\bar{q},e)$ are all linear in e, the $H_1^{ij}(q,\beta_q)$ must be independent of e.

The values of the $H_1^{ij}(q,\beta_q)$ for pseudoscalar, scalar, and vector bound states are listed in Tables II-C-3,4, and 5, respectively.

3) INTEGRATION OVER $d\beta_q$

The remaining integrals over $d\beta_q$ are all of the form

$$E_{ab}(q) \equiv \int_0^{\pi} d\beta_q \frac{\sin^a\beta_q \cos^b\beta_q}{(m^2+q^2-p^2)^2 + 4p^2q^2\cos^2\beta_q}, \tag{II-C-36}$$

where a and b are even integers. These integrals may be done by expanding the denominator in a power series in $\cos^2\beta_q$, and then using the formula[17]

$$\int_0^{\frac{\pi}{2}} \sin^{\mu-1}x \cos^{\nu-1}x \, dx = \frac{1}{2} B\left(\tfrac{1}{2}\mu, \tfrac{1}{2}\nu\right), \quad (\mathrm{Re}\,\mu > 0, \quad \mathrm{Re}\,\nu > 0) \tag{II-C-37}$$

where B(x,y) is the beta function. The result is

$$E_{ab}(q) = \frac{1}{(m^2+q^2-p^2)^2} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b+1}{2}+n\right)}{\Gamma\left(\frac{a+b}{2}+1+n\right)} x^n, \tag{II-C-38}$$

where

$$x = \frac{4p^2q^2}{(m^2+q^2-p^2)^2} \quad . \tag{II-C-39}$$

A closed form expression for $E_{ab}(q)$ can be obtained by expanding $(1+x)^{\alpha}$ in a power series, and comparing with eq. (II-C-38). (The motivation comes from calculating a few special cases of $E_{ab}(q)$ by contour integration.) It is found that if α is chosen as $(a-1)/2$, all but a finite number of terms in eq. (II-C-38) can be matched. The result is

$$E_{ab}(q) = \frac{1}{(m^2+q^2-p^2)^2} \frac{\pi(-1)^{b/2}}{x^{(a+b)/2}} \times \left\{ (1+x)^{(a-1)/2} - \sum_{n=0}^{\frac{a+b}{2}-1} \frac{\Gamma\left(\frac{a+1}{2}\right)}{n!\,\Gamma\left(\frac{a+1}{2}-n\right)} x^n \right\} . \tag{II-C-40}$$

The computer program used the power series expansion for small values of x and the above expression for large values of x.

The expressions for the $E_{ab}(q)$ which were used in the calculation are given in Table II-C-6.

Finally, the expressions for the $K^{ij}_{nn'}(q,k)$ which were used are listed in Tables II-C-7, II-C-8, and II-C-9, for pseudoscalar, scalar, and vector bound states respectively.

TABLE II-C-1

VALUES OF $I_{n'}^{m,s}\ (q,k,\beta_q)$

$$I_0^1 = 4\ (2\pi)^{3/2}\ \Delta_0$$

$$I_1^1 = 8\ (2\pi)^{3/2}\ \Delta_1\ \cos\beta_q$$

$$I_0^{(2,1)} = 4\ (2\pi)^{3/2}\ \frac{k}{q}\ \Delta_1$$

$$I_0^{(2,2)} = 0$$

$$I_1^{(2,1)} = 8\ (2\pi)^{3/2}\ \frac{k}{q}\ \Delta_2\ \cos\beta_q$$

$$I_1^{(2,2)} = 2\ (2\pi)^{3/2}\ \frac{k}{p}\ (\Delta_0 - \Delta_2)$$

$$I_0^{(3,1)} = 4\ (2\pi)^{3/2}\ \frac{k^2}{q^2}\ \Delta_2$$

$$I_0^{(3,2)} = (2\pi)^{3/2}\ k^2 (\Delta_0 - \Delta_2)$$

$$I_0^{(3,3)} = 0$$

where $\Delta_n \equiv \Delta_n\ (q,k)$.

TABLE II-C-2

VALUES OF $\tilde{H}_s^{ij}\ (q,k,\beta_q)$, $s \neq 1$

Pseudoscalar Bound States:

$$\tilde{H}_2^{i3} = \tilde{H}_1^{i2}$$

$$\tilde{H}_2^{i4} = 0$$

Scalar Bound States:

$$\tilde{H}_2^{i3} = \tilde{H}_1^{i2}$$

$$H_2^{i4} = 0$$

Vector Bound States:

$$\tilde{H}_2^{i1} = 0$$

$$\tilde{H}_2^{i2} = 0$$

$$\tilde{H}_2^{i3} = \tilde{H}_2^{i5}$$

$$\tilde{H}_3^{i3} = \tilde{H}_1^{i2}$$

$$\tilde{H}_4^{i3} = 0$$

$$\tilde{H}_2^{i4} = \tilde{H}_1^{i7}$$

$$\tilde{H}_3^{i4} = 0$$

$$\tilde{H}_4^{i4} = 0$$

$$\tilde{H}_2^{i6} = 0$$

$$\tilde{H}_2^{i8} = \tilde{H}_2^{i7}$$

TABLE II-C-3

$\tilde{H}_1^{ij}\ (q,\beta_q)$ FOR PSEUDOSCALAR BOUND STATES

$i \diagdown j$	1	2	3	4
1	$m^2+q^2+p^2$	$2mp^2$	$2mpq\cos\beta_q$	$4p^2q^2\sin^2\beta_q$
2	$2m$	$m^2-q^2+p^2$	$2pq\cos\beta_q$	$4mq^2$
3	0	$2pq\cos\beta_q$	$m^2+q^2-p^2$	$-4mpq\cos\beta_q$
4	-1	$-m$	0	$m^2-q^2-p^2$

TABLE II-C-4

$\tilde{H}_1^{ij}\ (q,\beta_q)$ FOR SCALAR BOUND STATES

$i \diagdown j$	1	2	3	4
1	$m^2-q^2-p^2$	$-2mpq\cos\beta_q$	$-2mq^2$	$2p^2q^2\sin^2\beta_q$
2	0	$m^2+q^2-p^2$	$-2pq\cos\beta_q$	$2mpq\cos\beta_q$
3	$2m$	$-2pq\cos\beta_q$	$m^2-q^2+p^2$	$-2mp^2$
4	-2	0	$-2m$	$m^2+q^2+p^2$

TABLE II-C-5

$\tilde{H}_1^{ij}(q,\beta_q)$ FOR VECTOR BOUND STATES

i \ j	1	2	3	4	5	6	7	8
1	$m^2-q^2-p^2$	$-2mpq\cos\beta$	$-2mq^2$	$2p^2q^2\sin^2\beta$	$-2m$	0	$-2p^2$	$-2pq\cos\beta$
2	0	$m^2+q^2-p^2$	$-2pq\cos\beta$	$2mpq\cos\beta$	0	$2pq\cos\beta$	0	0
3	$2m$	$-2pq\cos\beta$	$m^2-q^2+p^2$	$-2mp^2$	-2	$-2p^2$	0	0
4	-2	0	$-2m$	$m^2+q^2+p^2$	0	$2m$	-2	0
5	0	0	0	0	$m^2+q^2+p^2$	$2p^2q^2\sin^2\beta$	$2mp^2$	$2mpq\cos\beta$
6	0	0	0	0	-2	$m^2-q^2-p^2$	$-2m$	0
7	0	0	0	0	$2m$	$2mq^2$	$m^2-q^2+p^2$	$2pq\cos\beta$
8	0	0	0	0	0	$-2mpq\cos\beta$	$2pq\cos\beta$	$m^2+q^2-p^2$

TABLE II-C-6

VALUES OF E_{ab} (q)

$$E_{00}(q) = \frac{\pi g}{\sqrt{1+x}}$$

$$= \pi g \quad (1 - \frac{1}{2} x + \frac{3}{8} x^2 \ldots)$$

$$E_{02}(q) = \frac{\pi g}{x} \quad (1 - \frac{1}{\sqrt{1+x}})$$

$$= \frac{\pi g}{2} \quad (1 - \frac{3}{4} x + \frac{5}{8} x^2 \ldots)$$

$$E_{20}(q) = \frac{\pi g}{x} \quad (\sqrt{1+x} - 1)$$

$$= \frac{\pi g}{2} \quad (1 - \frac{1}{4} x + \frac{1}{8} x^2 \ldots)$$

$$E_{22}(q) = \frac{\pi g}{x^2} \quad (1 - \frac{1}{2} x - \sqrt{1+x})$$

$$= \frac{\pi g}{8} \quad (1 - \frac{1}{2} x + \frac{5}{16} x^2 \ldots)$$

$$E_{04}(q) = \frac{\pi g}{x^2} \quad (\frac{1}{\sqrt{1+x}} - 1 + \frac{1}{2} x)$$

$$= \frac{3\pi g}{8} \quad (1 - \frac{5}{6} x + \frac{35}{48} x^2 \ldots)$$

$$E_{42}(q) = \frac{\pi g}{x^3} \quad [1 + \frac{3}{2} x + \frac{3}{8} x^2 - (1+x)^{3/2}]$$

$$= \frac{\pi g}{16} \quad (1 - \frac{3}{8} x + \frac{3}{16} x^2 \ldots)$$

where

$$g = \frac{1}{(m^2+q^2-p^2)^2} \quad ,$$

and

$$x = \frac{4p^2q^2}{(m^2+q^2-p^2)^2} \quad .$$

TABLE II-C-7

VALUES OF $K^{ij}_{nn'}(q,k)$ FOR PSEUDOSCALAR BOUND STATES

i,n \ j,n'	1,0	2,0	3,1	4,0
1,0	$\frac{1}{\pi^3}(m^2+q^2+p^2) \times E_{20}\Delta_0 k^3$	$\frac{2mp^2}{\pi^3} E_{20}\Delta_0 k^3$	$\frac{mp}{\pi^3}[4E_{22}\Delta_2 + E_{20}(\Delta_0-\Delta_2)]k^4$	$\frac{4p^2q}{\pi^3} E_{40}\Delta_1 k^4$
2,0	$\frac{2m}{\pi^3} E_{20}\Delta_0 k^3$	$\frac{1}{\pi^3}(m^2-q^2+p^2) \times E_{20}\Delta_0 k^3$	$\frac{1}{\pi^3}[4pE_{22}\Delta_2 + \frac{m^2-q^2+p^2}{2p} \times E_{20}(\Delta_0-\Delta_2)]\, k^4$	$\frac{4mq}{\pi^3} E_{20}\Delta_1 k^4$
3,1	0	$\frac{4pq}{\pi^3} E_{22}\Delta_0 k^3$	$\frac{1}{\pi^3}[\,4\,\frac{m^2+q^2-p^2}{q} \times E_{22}\Delta_2 + 2qE_{22}(\Delta_0-\Delta_2)]\, k^4$	$\frac{-8mp}{\pi^3} E_{22}\Delta_1 k^4$
4,0	$-\frac{1}{\pi^3} E_{20}\Delta_0 k^3$	$-\frac{m}{\pi^3} E_{20}\Delta_0 k^3$	$-\frac{m}{2\pi^3 p} E_{20} \times (\Delta_0-\Delta_2)\, k^4$	$\frac{1}{\pi^3}\,\frac{m^2-q^2-p^2}{q} \times E_{20}\Delta_1 k^4$

TABLE II-C-8

VALUES OF $K^{ij}_{nn'}(q,k)$ FOR SCALAR BOUND STATES

i,n \ j,n'	1,1	2,0	3,1	4,1
1,1	$\frac{4}{\pi^3}(m^2-q^2-p^2)$ $\times E_{22}\Delta_1 k^3$	$-\frac{4}{\pi^3} mpqE_{22}\Delta_0 k^3$	$-\frac{2}{\pi^3}[4mqE_{22}\Delta_2$ $+mqE_{22}(\Delta_0-\Delta_2)]k^4$	$\frac{8}{\pi^3} p^2 q E_{42}\Delta_2 k^4$
2,0	0	$\frac{1}{\pi^3}(m^2+q^2-p^2)$ $\times E_{20}\Delta_0 k^3$	$\frac{1}{2\pi^3}[-8p E_{22}\Delta_2$ $+\frac{m^2+q^2-p^2}{p}$ $\times E_{20}(\Delta_0-\Delta_2)]k^4$	$\frac{4}{\pi^3} mp E_{22}\Delta_2 k^4$
3,1	$\frac{8}{\pi^3} mE_{22}\Delta_1 k^3$	$-\frac{4}{\pi^3} pq E_{22}\Delta_0 k^3$	$\frac{2}{\pi^3}[\frac{2(m^2-q^2+p^2)}{q}$ $\times E_{22}\Delta_2$ $-q E_{22}(\Delta_0-\Delta_2)]k^4$	$-\frac{4}{\pi^3}\frac{mp^2}{q}E_{22}\Delta_2 k^4$
4,1	$-\frac{8}{\pi^3} E_{22}\Delta_1 k^3$	0	$-\frac{8}{\pi^3}\frac{m}{q}E_{22}\Delta_2 k^4$	$\frac{4}{\pi^3}\frac{m^2+q^2+p^2}{q}$ $\times E_{22}\Delta_2 k^4$

TABLE II-C-9

VALUES OF $K^{ij}_{nn'}(q,k)$ FOR VECTOR BOUND STATES

$i,n \backslash j,n'$	1,0	2,1	3,0	4,0
1,0	$\frac{1}{\pi^3} \frac{m^2-q^2-p^2}{q}$ $\times E_{20}\Delta_1 k^4$	$-\frac{4}{\pi^3} mp E_{22}\Delta_2 k^4$	$-\frac{1}{2\pi^3} m E_{20} (3\Delta_2+\Delta_0)$ $\times k^5$	$\frac{1}{2\pi^3} p^2[E_{20}(\Delta_0-\Delta_2)$ $+ 4E_{40}\Delta_2] k^5$
2,1	0	$\frac{4}{\pi^3} \frac{m^2+q^2-p^2}{q}$ $\times E_{22}\Delta_2 k^4$	$-\frac{4}{\pi^3} \frac{p}{q} E_{22}\Delta_2 k^5$	$\frac{4}{\pi^3} \frac{mp}{q} E_{22}\Delta_2 k^5$
3,0	$\frac{2}{\pi^3} \frac{m}{q} E_{20}\Delta_1 k^4$	$-\frac{4}{\pi^3} p E_{22}\Delta_2 k^4$	$\frac{1}{2\pi^3} [E_{20}(\Delta_0-\Delta_2)$ $+2 \frac{m^2-q^2+p^2}{q^2}$ $\times E_{20}\Delta_2] k^5$	$-\frac{2}{\pi^3} \frac{mp^2}{q^2} E_{20}\Delta_2 k^5$
4,0	$-\frac{2}{\pi^3} \frac{1}{q} E_{20}\Delta_1 k^4$	0	$-\frac{2}{\pi^3} \frac{m}{q^2} E_{20}\Delta_2 k^5$	$\frac{1}{2\pi^3} E_{20} [(\Delta_0-\Delta_2)$ $+2 \frac{m^2+q^2+p^2}{q^2}\Delta_2] k^5$

TABLE II-C-9, CONTINUED

i,n \ j,n'	1,0	2,1	3,0	4,0
5,0	0	0	$\frac{1}{4\pi^3}(m^2+q^2+p^2)$ $\times E_{20}(\Delta_0-\Delta_2)k^5$	$-\frac{1}{2\pi^3} mp^2 E_{20}$ $\times(\Delta_0-\Delta_2)k^5$
6,0	0	0	$-\frac{1}{2\pi^3} E_{20}(\Delta_0-\Delta_2)k^5$	$\frac{1}{2\pi^3} m E_{20}$ $\times(\Delta_0-\Delta_2)k^5$
7,0	0	0	$\frac{1}{2\pi^3} m E_{20}$ $\times(\Delta_0-\Delta_2)k^5$	$-\frac{1}{4\pi^3}(m^2-q^2+p^2)$ $\times E_{20}(\Delta_0-\Delta_2)k^5$
8,1	0	0	0	$-\frac{1}{\pi^3} pq E_{22}$ $\times(\Delta_0-\Delta_2)k^5$

TABLE II-C-9, CONTINUED

j,n' / i,n	5,0	6,0	7,0	8,1
1,0	$-\frac{2}{\pi^3}\ m\,E_{20}\Delta_0\ k^3$	0	$-\frac{2}{\pi^3}\ p^2 E_{20}\Delta_0\ k^3$	$-\frac{1}{\pi^3}\ p\ [\ 4\ E_{22}\Delta_2$ $+\ E_{20}\ (\Delta_0-\Delta_2)]k^4$
2,1	0	$\frac{4}{\pi^3}\ p\ E_{22}\Delta_1\ k^4$	0	0
3,0	$-\frac{2}{\pi^3}\ E_{20}\Delta_0\ k^3$	$-\frac{2}{\pi^3}\ \frac{p^2}{q}\ E_{20}\Delta_1\ k^4$	0	0
4,0	0	$\frac{2}{\pi^3}\ \frac{m}{q}\ E_{20}\Delta_1\ k^4$	$-\frac{2}{\pi^3}\ E_{20}\Delta_0\ k^3$	$-\frac{1}{\pi^3}\ \frac{1}{p}\ E_{20}$ $\times\ (\Delta_0-\Delta_2)k^4$

TABLE II-C-9, CONTINUED

j,n' / i,n	5,0	6,0	7,0	8,1
5,0	$\frac{1}{\pi^3}(m^2+q^2+p^2) \times E_{20}\Delta_0 k^3$	$\frac{2}{\pi^3} p^2 q E_{40}\Delta_1 k^4$	$\frac{2}{\pi^3} mp^2 E_{20}\Delta_0 k^3$	$\frac{1}{\pi^3} mp [4E_{22}\Delta_2 + E_{20}(\Delta_0-\Delta_2)]k^4$
6,0	$-\frac{2}{\pi^3} E_{20}\Delta_0 k^3$	$\frac{1}{\pi^3}\frac{m^2-q^2-p^2}{q} \times E_{20}\Delta_1 k^4$	$-\frac{2}{\pi^3} m E_{20}\Delta_0 k^3$	$-\frac{1}{\pi^3}\frac{m}{p} E_{20} \times (\Delta_0-\Delta_2) k^4$
7,0	$\frac{2}{\pi^3} m E_{20}\Delta_0 k^3$	$\frac{2}{\pi^3} mq E_{20}\Delta_1 k^4$	$\frac{1}{\pi^3}(m^2-q^2+p^2) \times E_{20}\Delta_0 k^3$	$\frac{1}{\pi^3}[4p E_{22}\Delta_2 + \frac{m^2-q^2+p^2}{2p} \times E_{20}(\Delta_0-\Delta_2)]k^4$
8,1	0	$-\frac{4}{\pi^3} mp E_{22}\Delta_1 k^4$	$\frac{4}{\pi^3} pq E_{22}\Delta_0 k^3$	$\frac{1}{\pi^3}[4\frac{m^2+q^2-p^2}{q} \times E_{22}\Delta_2 + 2q E_{22}(\Delta_0-\Delta_2)]k^4$

D. TRUNCATION OF THE EXPANSION

The computer calculations have been carried out using only the lowest nonvanishing term in the expansion of each invariant function in the $C^{o}_{n}(\beta_q)$. With this truncation, it is possible to accurately calculate many properties of the deeply bound states. The inaccuracies introduced by the truncation depend on the specific case, so they will be discussed one by one.

For pseudoscalar bound states, solutions of two types have been found. For both scalar and vector interactions, the B-S wavefunctions that were studied were dominated by $\chi^{(1)}_{o}(q)\gamma_5$. In the limit of $\bar{p}\rightarrow 0$, these wavefunctions are invariant under rotations of $\bar{q}$, which means that the states are O(4) singlets. For $\bar{p} = 0$, the equation for this term decouples completely from all other terms in the expansion. Thus, the truncation allows an exact calculation of the value of g^2 corresponding to $\bar{p} = 0$. Assuming the dominance of this term, one can see from the $K^{ij}_{nn'}(q,k)$ that

$$\begin{aligned} \chi^{(2)}_{o}(q) &\sim \chi^{(1)}_{o}(q) \\ \chi^{(3)}_{1}(q) &\sim p\,\chi^{(1)}_{o}(q) \\ \chi^{(4)}_{o}(q) &\sim \chi^{(1)}_{o}(q)\,. \end{aligned} \tag{II-D-1}$$

(Note that $M^{(2)} \sim M^{(4)} \sim p$, so each of the three terms gives a correction to the wavefunction which is $\sim p$.) Referring again to the $K^{ij}_{nn'}(q,k)$, it can be seen that each of these terms leads to a correction in $\chi^{(1)}_{o}(q)$ which is $\sim p^2$. We can now calculate the order of magnitude of the next term in the $C^{o}_{n}(\beta_q)$ expansion. Notice, from eq. (II-B-17), that it is easy to obtain $K^{ij}_{n+2,n'}(q,k)$ from $K^{ij}_{nn'}(q,k)$. One merely replaces $C^{o}_{n}(\beta_q)$ by $C^{o}_{n+2}(\beta_q)$.

In particular, suppose $K^{ij}_{o,n'}(q,k) \propto E_{20}(q)$. It follows that $K^{ij}_{2,n'}(q,k) \propto (4E_{22}(q) - E_{20}(q)) \sim p^2 E_{20}$, so $K^{ij}_{2,n'}(q,k) \sim p^2 K^{ij}_{o,n'}(q,k)$. Similarly, if $K^{ij}_{1,n'}(q,k) \propto E_{22}(q)$, then $K^{ij}_{3,n'}(q,k) \sim p^2\ K^{ij}_{1,n'}(q,k)$. Using these relations, it can be seen that the next terms in the expansion are of order

$$
\begin{aligned}
\chi_2^{(1)}(q) &\sim p^2\ \chi_0^{(1)}(q) \\
\chi_2^{(2)}(q) &\sim p^2\ \chi_0^{(2)}(q) \\
\chi_3^{(3)}(q) &\sim p^2\ \chi_1^{(3)}(q) \\
\chi_2^{(4)}(q) &\sim p^2\ \chi_0^{(4)}(q)\,.
\end{aligned}
\qquad \text{(II-D-2)}
$$

The neglected terms are all overshadowed by the leading terms in the O(4) limit, and so these terms will not contribute to the normalization integral or to the integrals for other matrix elements or perturbation expressions. (It is conceivable that the second term will give a nonnegligible contribution in cases where the contribution of the leading term integrates to zero due to a symmetry which does not nullify the contribution of the second term. However, no such cases were found.) Finally, we can calculate the order of the corrections in $\chi_o^{(1)}(q)$ resulting from these terms. The contributions of the last three terms are $\sim p^4$. One can calculate that $K^{11}_{0,2}(q,k) \sim p^2$, so the neglected corrections to $\chi_o^{(1)}(q)$ are all $\sim p^4$. Since the equation for $\chi_o^{(1)}(q)$ is correct to order p^2, it follows that p^2 is determined correctly, up to corrections of relative order p^2/m^2.

When the equations were solved for pseudoscalar bound states with a pseudoscalar interaction, a second type of solution was found. These wavefunctions were dominated by the last three terms of the expansion in Lorentz invariant functions. In the O(4) limit, the equations for these three terms decouple from all other terms, and so the value of g^2 corresponding to $p = 0$ is found exactly. In this limit, the state is degenerate with a state of $J^{PC} = 1^{++}$, forming an O(4) quartet which transforms as a four vector (otherwise known as the $(\frac{1}{2},\frac{1}{2})$ representation). By an analysis similar to the one above, it is found that the previous success is not repeated. The dominant contribution of the wavefunction comes from

$$p\,\chi_0^{(2)}(q) \sim \chi_1^{(3)}(q) \sim p\,\chi_0^{(4)}(q)\,. \tag{II-D-3}$$

The calculation also includes the term

$$\chi_0^{(1)}(q) \sim p^2\,\chi_0^{(2)}(q)\,. \tag{II-D-4}$$

The first neglected terms in the expansions are of order

$$\begin{aligned}\chi_2^{(1)}(q) &\sim \chi_0^{(1)}(q)\\ \chi_2^{(2)}(q) &\sim p^2\,\chi_0^{(2)}(q)\\ \chi_3^{(3)}(q) &\sim p^2\,\chi_1^{(3)}(q)\\ \chi_2^{(4)}(q) &\sim p^2\,\chi_0^{(4)}(q)\,.\end{aligned} \tag{II-D-5}$$

The neglected terms $\chi_2^{(2)}(q)$, $\chi_3^{(3)}(q)$, and $\chi_2^{(4)}(q)$ will lead to corrections to the dominant part of the wavefunction of relative order p^2, which means that p^2 will not be found correctly as it was in the previous case. Furthermore, the neglected term $\chi_2^{(1)}(q)$ contributes nonnegligibly to the normalization integral in the O(4) limit, so no matrix elements can be correctly computed. For these cases, the only accurate result is the value of g^2 for $p=0$.

There are presumably other types of solutions transforming according to higher representations of O(4), but these solutions cannot be found without including more terms in the expansion.

Next, consider the vector bound states. Only one type of solution has been found, corresponding to an O(4) quartet. The wave-function is dominated by the terms

$$i\,\chi_0^{(1)}(q)\;\bar{q}\cdot e$$
$$\chi_0^{(3)}(q)\;\bar{q}\cdot e\;\gamma\cdot\bar{q}$$

and

$$\chi_0^{(5)}(q)\;\gamma\cdot e\,,$$

all of the same order. The other terms are of order

$$\begin{aligned}
\chi_1^{(2)}(q) &\sim p\,\chi_0^{(1)}(q)\\
\chi_0^{(4)}(q) &\sim \chi_0^{(1)}(q)\\
\chi_0^{(6)}(q) &\sim \chi_0^{(1)}(q)\\
\chi_0^{(7)}(q) &\sim \chi_0^{(1)}(q)\\
\chi_1^{(8)}(q) &\sim p\,\chi_0^{(1)}(q)\,.
\end{aligned} \qquad \text{(II-D-6)}$$

For each of the eight components,

$$\chi_{n_0+2}^{(i)}(q) \sim p^2\,\chi_{n_0}^{(i)}(q), \qquad \text{(II-D-7)}$$

where n_o is the first nonvanishing term. Thus, the normalization integral and all the matrix elements and perturbation expressions can be calculated in the O(4) limit. The neglected terms $\chi_2^{(1)}(q)$ and $\chi_2^{(3)}(q)$ give contributions to the dominant part of the wavefunction of relative order p^2, which means that the value of p^2 obtained directly from the solution will not be correct. However, the perturbation expression (I-P-18) can be calculated, resulting in a value for p^2 which is correct to relative order p^2/m^2.

Finally, consider the scalar bound states. The only solutions found were the O(4) partners, or daughters, of the vector bound states. (In the O(4) limit, these solutions may be obtained from the vector solutions by replacing the polarization vector e_μ by $\hat{p}_\mu$.) The wavefunction is dominated by the components

$$\chi_1^{(1)}(q) \sim p\,\chi_0^{(2)}(q) \sim \chi_1^{(3)}(q) \qquad \text{(II-D-8)}$$

with

$$\chi_1^{(4)}(q) \sim \chi_1^{(1)}(q)\ . \qquad \text{(II-D-9)}$$

Each of the four terms obeys eq. (II-D-7), so the normalization integral, matrix elements, and perturbation expressions can all be calculated in the O(4) limit. The neglected terms $\chi_2^{(2)}(q)$ and $\chi_3^{(3)}(q)$ produce corrections to the dominant part of the wavefunction of relative order p^2, so the value of p^2 obtained directly will be incorrect. However, the perturbation expression (I-P-18) can again be used, with the same accuracy as before.

E. APPROXIMATION BY A MATRIX EQUATION

To solve the problem numerically, the integral equation (II-B-16) is converted to a matrix equation by the techniques of numerical integration.

The first step is to transform the range of integration to a finite interval. This may be done by the change of variables

$$k(x) = M \frac{1+x}{1-x} ,$$
$$x(k) = \frac{k-M}{k+M} , \qquad \text{(II-E-1)}$$

where M is a scaling mass chosen to match the interval over which $\chi(q)$ is significant. The truncated integral equation then becomes

$$\chi^{(i)}(q) = g^2 \sum_j \int_{-1}^{1} dx \frac{(k+M)^2}{2M} K^{ij}(q,k) \chi^{(j)}(k), \qquad \text{(II-E-2)}$$

where the indices n and n' have been omitted, since they take on only one value for each i or j.

This integral equation can be converted to a matrix equation by any of a number of techniques of numerical integration. One accurate technique is Gaussian quadrature[18]:

$$\int_{-1}^{1} f(x)\,dx \approx \sum_{\ell=1}^{n} w_\ell f(x_\ell), \qquad \text{(II-E-3)}$$

where the x_ℓ's are the zeros of the n'th order Legendre polynomial $P_n(x)$, and

$$w_\ell = \frac{2(1-x_\ell^2)}{n^2[P_{n-1}(x_\ell)]^2} . \qquad \text{(II-E-4)}$$

This integration technique is exact if f(x) is a polynomial of degree $\leq$ 2n + 1.

Gaussian quadrature was tried, and it resulted in solutions which were stable to variations of n and M, provided that the masses μ and Λ appearing in the interaction function (eqs. (II-A-4) or (II-A-6))were not too small compared to m, the quark mass. However, when μ and Λ are both small, the function $\Delta_n(q,k)$ (defined by eq. (II-C-12)) becomes very sharply peaked in the neighborhood of $k \approx q$. The integrand is therefore sharply peaked, and Gaussian quadrature requires more integration points than the computer capacity permitted. The program used up to 24 points per invariant function for spin zero bound states, and up to 12 points per invariant function for vector bound states.

The problem can be circumvented by the technique of polynomial interpolation with an arbitrary weight function. Consider the integral

$$I = \int_{-1}^{1} dx\, g(x)\, f(x) , \qquad \text{(II-E-5)}$$

where g(x) is a fixed weight function and f(x) is arbitrary. By approximating f(x) by the (n-1)'th degree polynomial which fits f(x) at the points $x_1,\ldots,x_n$, one obtains

$$I \approx \sum_{\ell=1}^{n} w_\ell\, f(x_\ell) , \qquad \text{(II-E-6)}$$

where

$$w_\ell = \int_{-1}^{1} dx\, g(x) \frac{p(x)}{(x - x_\ell)\, p'(x_\ell)} \tag{II-E-7}$$

with

$$p(x) \equiv \prod_{\ell=1}^{n} (x - x_\ell)\,. \tag{II-E-8}$$

This technique will be accurate if f(x) is a smoothly varying function, regardless of how sharply peaked g(x) is.

By looking at Tables II-C-7, 8, and 9 for the $K^{ij}(q,k)$, one sees that they can all be written in the form

$$K^{ij}(q,k) = \left[F^{ij}(q)\, \Delta_j(q,k) + F'^{ij}(q)\, \Delta'_j(q,k) \right] k^{n(j)}\,, \tag{II-E-9}$$

where

$$\Delta_j\,,\ \Delta'_j = \Delta_0,\ \Delta_1,\ \Delta_2,\ \text{or}\ (\Delta_0 - \Delta_2)\,, \tag{II-E-10}$$

and

$$n(j) = 3,\ 4,\ \text{or}\ 5\,. \tag{II-E-11}$$

By looking at the wavefunctions obtained using Gaussian quadrature, it is found that $k^{n(j)}\chi^{(j)}(k)$ is a smooth function which can be used as the $f(x)$ for the polynomial interpolating integration. The other factors are all incorporated into $g(x)$. The resulting matrix equation is then

$$\chi^{(i)}(q_m) = g^2 \sum_j \sum_{\ell=1}^{n} \mathcal{M}_{mi,\ell j}\, \chi^{(j)}(q_\ell), \tag{II-E-12}$$

where

$$\mathcal{M}_{mi,\ell j} = q_\ell^{n(j)} \int_{-1}^{1} dx\, \frac{(k+M)^2}{2M} \Big[F^{ij}(q_m)\,\Delta_j(q_m,k) + F'^{ij}(q_m)\,\Delta'_j(q_m,k)\Big] \frac{P(x)}{(x-x_\ell)\,P'(x_\ell)} . \tag{II-E-13}$$

The integration points, $x_1,\ldots,x_n$, are still free to be chosen. It is convenient to choose these points as the zeros of $P_n(x)$, as they would be for Gaussian quadrature. Then once the values of the wavefunctions on these mesh points have been found, one can conveniently use Gaussian quadrature to calculate the normalization integral, matrix elements, and perturbation expressions.

The integration indicated in the previous equation cannot be done in closed form, but it can be done numerically. The integrand is sharply peaked, so one must be careful. The range of integration was divided into three segments, with the center segment just covering the peak. Each segment was then integrated using 64 point Gaussian quadrature.

F. NUMERICAL SOLUTION OF THE PROBLEM

1. DIAGONALIZATION OF THE MATRIX

In most cases, it was found that the desired solution to the matrix equation (II-E-12) could be obtained by iteration. In general one would expect the iteration to converge to the eigenvector corresponding to the eigenvalue of largest absolute value, which means the smallest value of $|g^2|$. This vector will usually be the ground state. Sometimes, however, the process converges to spurious solutions of negative g^2. This problem can be remedied by iterating the matrix $(\mathcal{M} + \lambda I)$, where λ is chosen large enough so that the absolute value of the desired positive eigenvalue exceeds that of the unwanted negative eigenvalue.

When convergence occurred, it usually occurred within 100 iterations. The program used the very stringent convergence requirement that a single iteration should reproduce each component of the eigenvector to within one part in 10^5.

In some cases the iteration failed to converge, and then the problem was solved by the slower method of searching for zeros of the characteristic determinant. The program then solved for the eigenvector, and tested it by a single iteration. The significant parts of the wavefunction typically checked to better than one part in 10^5, although there was typically a component at the tail of the wavefunction which failed by order unity to be reproduced. Sometimes these eigenvectors were improved by iteration.

2. INTEGRATION OVER THE WAVEFUNCTIONS

To calculate the normalization integral and to calculate matrix elements and perturbation expressions, it is necessary to perform integrations of the form

$$\int d^4\bar{q}\, \mathrm{Tr}\left\{ \bar{\chi}(\bar{p},\bar{q},e)\, \Gamma_1\, \chi(p,q,e)\, \Gamma_2 \right\},$$

where Γ_1 and Γ_2 represent matrices which may depend upon $\bar{q}$. These expressions can be reduced to expressions involving only the Lorentz invariant expansion functions, but it is easy enough for the computer to calculate the expression in its original form. The 4 x 4 matrices χ and $\bar{\chi}$ were constructed from the invariant functions, and the integrand was evaluated by multiplying the matrices and taking the trace at the end.

The radial integration was done by Gaussian quadrature, using the mesh points on which the wavefunctions were computed. The angular integrations are of the form

$$\int_0^{\pi} \sin^2\beta\, d\beta \times (\text{polynomial in } \cos\beta)$$

$$\int_0^{\pi} \sin\theta\, d\theta \times (\text{polynomial in } \cos\theta)$$

$$\int_0^{2\pi} d\phi \times (\text{polynomial in } \sin\phi,\ \cos\phi).$$

For the maximum degree polynomials which appear in these calculations, these integrations can be done exactly by using four point Gaussian - Gegenbauer integration for the first, four point Gaussian integration (also known as Gaussian - Legendre) for the second, and six point equal spacing equal weight integration for the third.

3. CONSISTENCY TESTS

Using the perturbation expressions (I-P-12) and (I-P-18), one can calculate

$$\left(\frac{\partial M_B^2}{\partial m}\right)_{g^2} \quad \text{and} \quad \left(\frac{\partial M_B^2}{\partial g^2}\right)_m .$$

When using the computer program, it is most convenient to take m and M_B as independent variables, and so the easiest partial derivatives to check are

$$\left(\frac{\partial g^2}{\partial m}\right)_{M_B} = -\left(\frac{\partial M_B^2}{\partial m}\right)_{g^2} \Bigg/ \left(\frac{\partial M_B^2}{\partial g^2}\right)_m \qquad \text{(II-F-1)}$$

and

$$\left(\frac{\partial g^2}{\partial M_B}\right)_m = 1 \Bigg/ \left(\frac{\partial M_B^2}{\partial g^2}\right)_m . \qquad \text{(II-F-2)}$$

The first of the above derivatives was checked, by making small variations in m, for pseudoscalar and vector bound states, with each of the three types of interactions. In all six cases there was better than 1% agreement between the actual value of Δg^2 and the value calculated from the perturbation expression.

The second partial derivative above can be checked in only two cases -- a pseudoscalar bound state with either a scalar or vector interaction. For these two cases, the agreement was again better than 1%. As discussed in section II-D, in the other cases the direct calculation (the variation of M_B) fails, while the perturbation calculation remains valid.

4. NUMERICAL ACCURACY

By varying the method of calculation (e.g., the number of integration points and the scaling mass M), it is possible to obtain a rough estimate of the accuracy of the results. It appears that all the results are accurate to about 1% or better.

PART III:

DISCUSSION OF THE RESULTS

A. GENERAL PROPERTIES OF THE SOLUTIONS

1. WAVEFUNCTIONS

Using the singly regulated interaction, the wavefunctions are found to give internal quark momenta which are comparable to the quark mass. An example of these wavefunctions is shown in Figure III-A-1, at the end of this section. These wavefunctions describe a pseudoscalar bound state resulting from a scalar interaction with the following choice of parameters:

m (quark mass) = 1.0

μ (inverse range of attractive interaction) = 0.05

Λ (inverse range of repulsive interaction) = 0.10

M_B (bound state mass) = 0.01.

Using the doubly regulated form of the interaction, one can produce bound states with internal quark momenta which are small compared to the quark mass. In fact, the B-S equation remains an equation of the Fredholm type in the limit as $m \longrightarrow \infty$, so one expects that for large m, the internal quark momenta are governed by μ and Λ only. Unfortunately the computer solutions become unstable for $m \gtrsim 20\Lambda$, so this limit of very large m has not been investigated.

Examples of wavefunctions obtained using doubly regulated wavefunctions are shown in Figures III-A-2 and 3, for a pseudoscalar and vector bound state, respectively. Both cases result from a scalar interaction, with the same choice of parameters as above.

To discuss the spinor structure, it is convenient to divide the 4×4 matrix χ into four 2×2 matrices:

$$\chi = \begin{pmatrix} \chi_{+-} & \chi_{++} \\ \chi_{--} & \chi_{-+} \end{pmatrix}$$

For a nonrelativistic bound state, χ_{++} dominates over the other components.

For the pseudoscalar bound states, two types of solutions have been found. For scalar and vector interactions, the most deeply bound states are O(4) singlets, dominated by the term proportional to γ_5. Thus $\chi_{++} \approx \chi_{--}$. For a pseudoscalar interaction the most deeply bound pseudoscalar states belong to O(4) quartets, and are dominated by the last three terms of the expansion. In this case, $\chi_{++} \approx -\chi_{--}$. So in neither case is the spinor structure nonrelativistic.

All of the vector bound states which were found belong to O(4) quartets, and are dominated by the first, third, and fifth terms of the expansion. For these states $\chi_{++} \approx -\chi_{--}$, and again the spinor structure is highly relativistic.

2. QUADRATIC MASS FORMULA

One general result of the calculations is the justification of the successful quadratic form of the Gell-Mann-Okubo mass formula for mesons.

Eq. (I-P-12) for $\partial M_B^2/\partial m_i$ suggests that M_B^2 should vary linearly with m_i, but this suggestion is not conclusive. In fact one can show that in the nonrelativistic limit, the equation reduces to the linear relation $\partial M_B/\partial m_i = 1$. In general, the equation will lead to a linear mass formula if the right hand side is proportional to M_B, and a quadratic mass formula if the right hand side is independent of M_B. For all of the cases calculated, the right hand side approaches a finite value in the O(4) limit $M_B \longrightarrow 0$, leading to a quadratic mass formula near $M_B = 0$.

Since the method we have used to solve the B-S equation is valid only in the neighborhood $M_B \approx 0$, it is not possible to use the solutions to determine the range of validity of the quadratic mass formula. Presumably the formula holds whenever M_B is small compared to both the quark mass and the quark momentum.

3. THE RATIO f_π/f_K AND THE VAN ROYEN - WEISSKOPF PARADOX

Another general result of the calculations is that, in contrast to the nonrelativistic model, the approximate equality of f_π and f_K arises naturally.

It was pointed out by Van Royen and Weisskopf[20] that this relation is difficult to reconcile with the nonrelativistic quark model. It can be explained only by assuming that the wavefunctions $\psi(x)$ for the π and the K show very large SU(3) symmetry breaking. The value of these wavefunctions at the origin must obey the relation

$$\frac{\psi^{\pi}(0)}{\psi^{K}(0)} \approx \sqrt{\frac{M_\pi}{M_K}} \tag{III-A-1}$$

In the relativistic model, f_P (f_π or f_K) is given by eq. (I-L-9). In terms of the expansion functions introduced in section II-B, this relation can be written as

$$f_P = \frac{2}{(2\pi)^{5/2}} \int_0^\infty q^3\,dq\,\left[\chi_0^{(2)}(q) + \frac{q}{M_B}\,\chi_1^{(3)}(q)\right], \tag{III-A-2}$$

where as usual we have kept only the leading term in the expansion of each Lorentz invariant function. Since the π and the K are both deeply bound, we want to know how this expression for f_P behaves in the region of $M_B \approx 0$.

The relative magnitudes of the expansion functions were discussed in section II-D. For O(4) singlet states, the relative magnitudes are given by

$$\chi_0^{(1)}(q) \sim \chi_0^{(2)}(q) \sim \chi_0^{(4)}(q)$$

$$\chi_1^{(3)}(q) \sim M_B\, \chi_0^{(1)}(q)\,. \qquad \text{(III-A-3)}$$

The wavefunction is dominated by $\chi_0^{(1)}(q)\,\gamma_5$, with the other terms giving corrections proportional to M_B. Llewellyn Smith[21] has shown that wavefunctions with these order of magnitude relations, which he calls Model 1, give the correct ratio of f_π/f_K. For completeness, the argument is repeated here.

To find how the absolute magnitudes of these terms depend upon M_B as $M_B \longrightarrow 0$, one must look at the normalization condition, eq. (I-J-13). In terms of the expansion functions, the normalization condition can be written as

$$\sum_{\substack{ij \\ i\leq j}} \int_0^\infty q^3\, dq\; N^{ij}(q)\, \chi^{(i)}(q)\, \chi^{(j)}(q) = 1\,, \qquad \text{(III-A-4)}$$

where only the first nonvanishing term of the expansion of each invariant function has been included. The nonzero $N^{ij}(q)$ are given by

$$\begin{aligned}
N^{11} &= -\pi \\
N^{12} &= 2\pi m \\
N^{13} &= \pi \frac{mq}{p} \\
N^{14} &= -3\pi q^2 \\
N^{22} &= -\pi p^2 \\
N^{23} &= -\pi p q \\
N^{44} &= -3\pi p^2 q^2\,.
\end{aligned} \qquad \text{(III-A-5)}$$

So for O(4) singlets, the normalization condition is dominated by the first four of these terms. The dependence of each function on M_B is then given by

$$\chi_0^{(1)} \sim \chi_0^{(2)} \sim \chi_0^{(4)} \sim \text{constant}$$
$$\chi_1^{(3)} \sim M_B . \tag{III-A-6}$$

Looking at the relation for f_P, one concludes that f_P is independent of M_B in the deeply bound region; so $f_\pi \approx f_K$.

For pseudoscalar bound states which belong to O(4) quartets, the situation is quite different. The method of analysis is the same, except that the contribution of $\chi_2^{(1)}(q)$ to the normalization integral cannot be ignored. The result is that

$$\chi_0^{(2)} \sim \chi_0^{(4)} \sim \frac{1}{M_B}$$
$$\chi_1^{(3)} \sim \text{constant} \tag{III-A-7}$$
$$\chi_0^{(1)} \sim \chi_2^{(1)} \sim M_B .$$

These relations correspond to Model 2 of Llewellyn Smith. It follows that $f_P \sim 1/M_B$, so one would predict that

$$\frac{f_\pi}{f_K} \approx \frac{M_K}{M_\pi} .$$

Thus, the ratio of f_π to f_k is given correctly by the relativistic model, provided that the pseudoscalar mesons are O(4) singlets. For either a scalar or a vector interaction, the ground states were always found to be of this form. For a pseudoscalar interaction, the ground state was always found to be a quartet.

4. THE RATIO $g_\rho : g_\omega : g_\phi$

Another general result of the calculations is the prediction of the ratio $g_\rho : g_\omega : g_\phi$, the decay constants for the electromagnetic leptonic decays of the vector mesons ($V \to \ell^+ \ell^-$) (see section I-M). In this case the prediction is at odds with experiment, but the discrepancies are not large. The mass ratios of the vector mesons are not far enough from unity to give a clear cut test.

As discussed in section I-M, the constants g_V are related to constants $\bar{g}_V$, which are related to the B-S wavefunction by eq. (I-M-15). In terms of the expansion functions, this relation is

$$\bar{g}_V = - \frac{e}{(2\pi)^{5/2} M_B} \int_0^\infty q^3 \, dq \left[4 \chi_0^{(5)}(q) + q^2 \chi_0^{(3)}(q) \right], \qquad \text{(III-A-8)}$$

where only the leading terms have been kept. All of the ground states which were found belong to O(4) quartets (four-vectors), and the expansion functions have relative magnitudes discussed in section II-D. Using the normalization condition to determine the absolute magnitudes, one finds that the expansion functions depend on M_B as follows:

$$\begin{gathered} \chi_0^{(1)} \sim \chi_0^{(3)} \sim \chi_0^{(4)} \sim \chi_0^{(5)} \sim \chi_0^{(6)} \sim \chi_0^{(7)} \sim \text{constant} \\ \chi_1^{(2)} \sim \chi_1^{(8)} \sim M_B \end{gathered} \qquad \text{(III-A-9)}$$

This corresponds to Llewellyn Smith's Model 2 for vector mesons, and clearly predicts that

$$\bar{g}_V \sim \frac{1}{M_V} \qquad \text{(III-A-10)}$$

for deeply bound states.

Table III-A-1 lists the values of g_V and the values of $M_V g_V$, for the ρ, ω, and ϕ decaying into the modes e^+e^- and $\mu^+\mu^-$. The values have been calculated from the widths and branching ratios in the 1971 listings of the Particle Data Group[11]. If one accepts the quoted experimental errors, then the computed values of chi-square show that it is consistent with experiment to believe that all the values of g_V are equal. However, there is only a probability of about .005 that chi-square for the values of M_V g_V would be so large, if their true values were equal.

Nonetheless, this discrepancy does not appear to me to be a serious deficiency of the model. All of the values of M_V g_V lie within 15% of a central value, and perhaps that is all that can be expected in so simple a model. Furthermore, the experiments could be wrong. The whole argument hinges on the ϕ-meson measurements, and the values measured from e^+e^- and $\mu^+\mu^-$ decay are inconsistent. The actual inconsistency is greater than it appears in Table III-A-1, because the uncertainty in the ϕ width contributes significantly to that of g_ϕ. The branching ratio measurements, which by μ-e universality should be equal up to a negligible phase space correction, are listed in the Particle Data Group tables as

$$\phi \to e^+e^- : \quad .035 \pm .003$$

$$\phi \to \mu^+\mu^- : \quad .023 \pm .005.$$

If one insists that the model predict $\bar{g}_\rho \approx \bar{g}_\omega \approx \bar{g}_\phi$, then one must try to arrange for the bound state wavefunction to have the form of Llewellyn Smith's Model 1. The state would have to belong to an O(4) sextet (antisymmetric rank 2 tensor, containing a 1^- and 1^+ particle), and O(4) nonet (traceless symmetric rank 2 tensor, containing 2^+, 1^-, and 0^+), or something more complicated. It is not clear what modifications of the interaction are necessary to give a ground state of this form, but it may be possible.

TABLE III-A-1

VALUES OF $\bar{g}_V$ and $M_V\bar{g}_V$

$\bar{g}_V$:

$\bar{g}_\rho\ (e^+e^-) = 69 \pm 7$ MeV

$\bar{g}_\rho\ (\mu^+\mu^-) = 73 \pm 17$ MeV

$\bar{g}_\omega\ (e^+e^-) = 66 \pm 9$ MeV*

$\bar{g}_\phi\ (e^+e^-) = 73 \pm 4$ MeV

$\bar{g}_\phi\ (\mu^+\mu^-) = 59 \pm 7$ MeV

Chi-square = 3.2.

$M_V\bar{g}_V$:

$M_\rho\bar{g}_\rho\ (e^+e^-) = (5.3 \pm 0.5) \times 10^4\ \mathrm{MeV}^2$

$M_\rho\bar{g}_\rho\ (\mu^+\mu^-) = (5.6 \pm 1.3) \times 10^4\ \mathrm{MeV}^2$

$M_\omega\bar{g}_\omega\ (e^+e^-) = (5.2 \pm 0.7) \times 10^4\ \mathrm{MeV}^2$ *

$M_\phi\bar{g}_\phi\ (e^+e^-) = (7.4 \pm 0.4) \times 10^4\ \mathrm{MeV}^2$

$M_\phi g_\phi\ (\mu^+\mu^-) = (6.0 \pm 0.7) \times 10^4\ \mathrm{MeV}^2$

Chi-square = 14.5.

*Error includes a scale factor, from the Particle Data Group tables, of 1.4.

FIGURE III-A-1

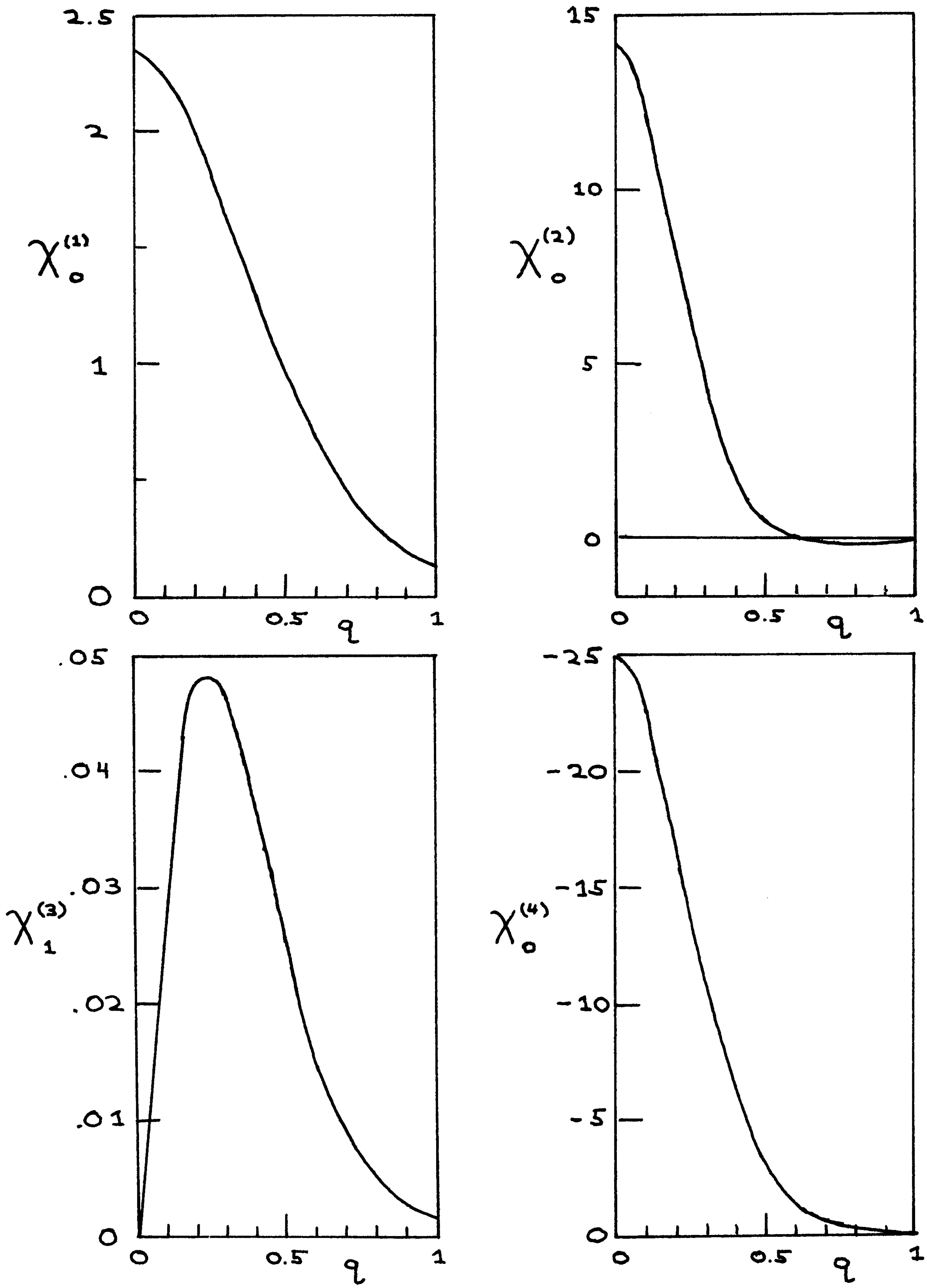


FIGURE III-A-2

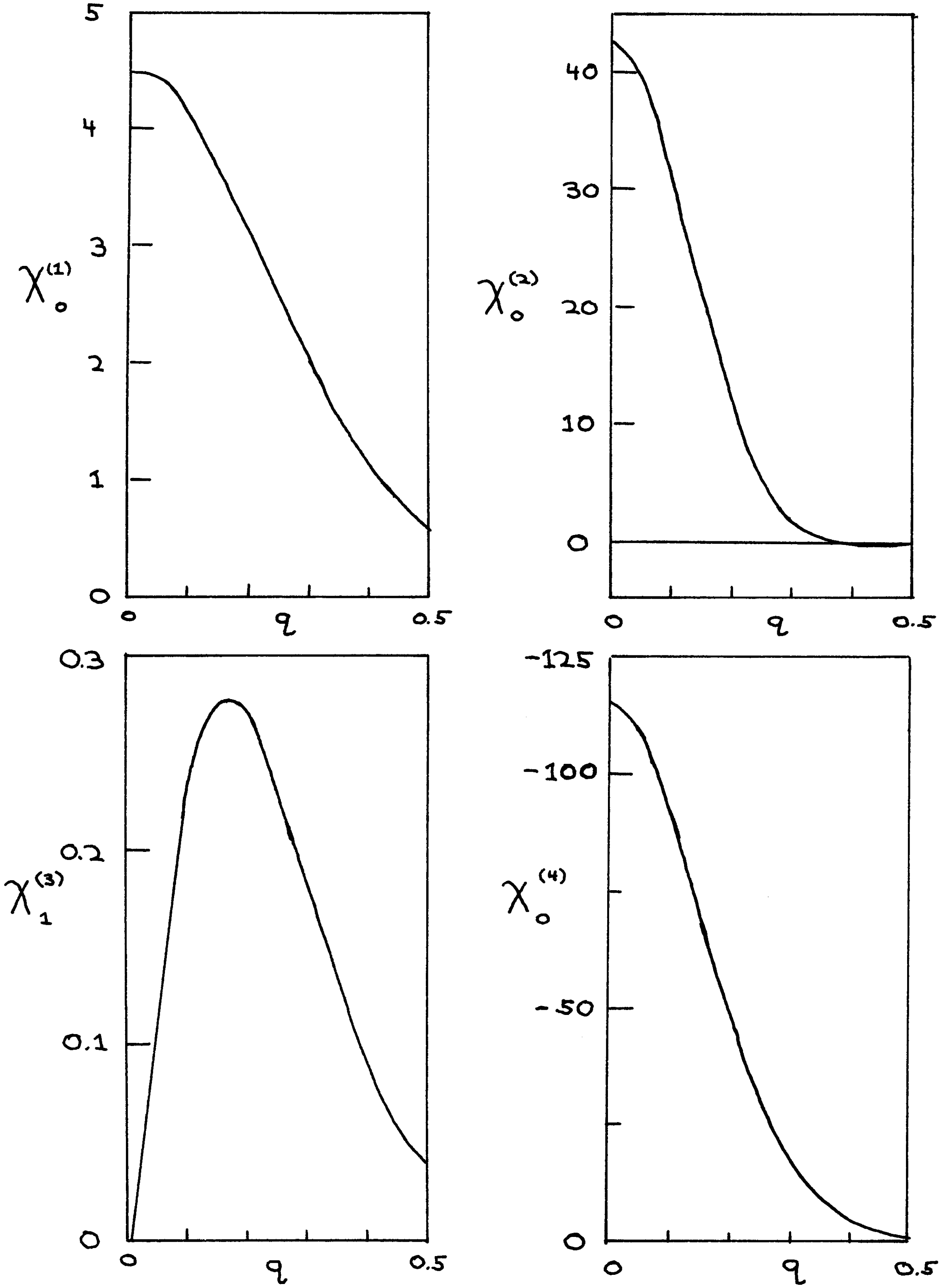


FIGURE III-A-3

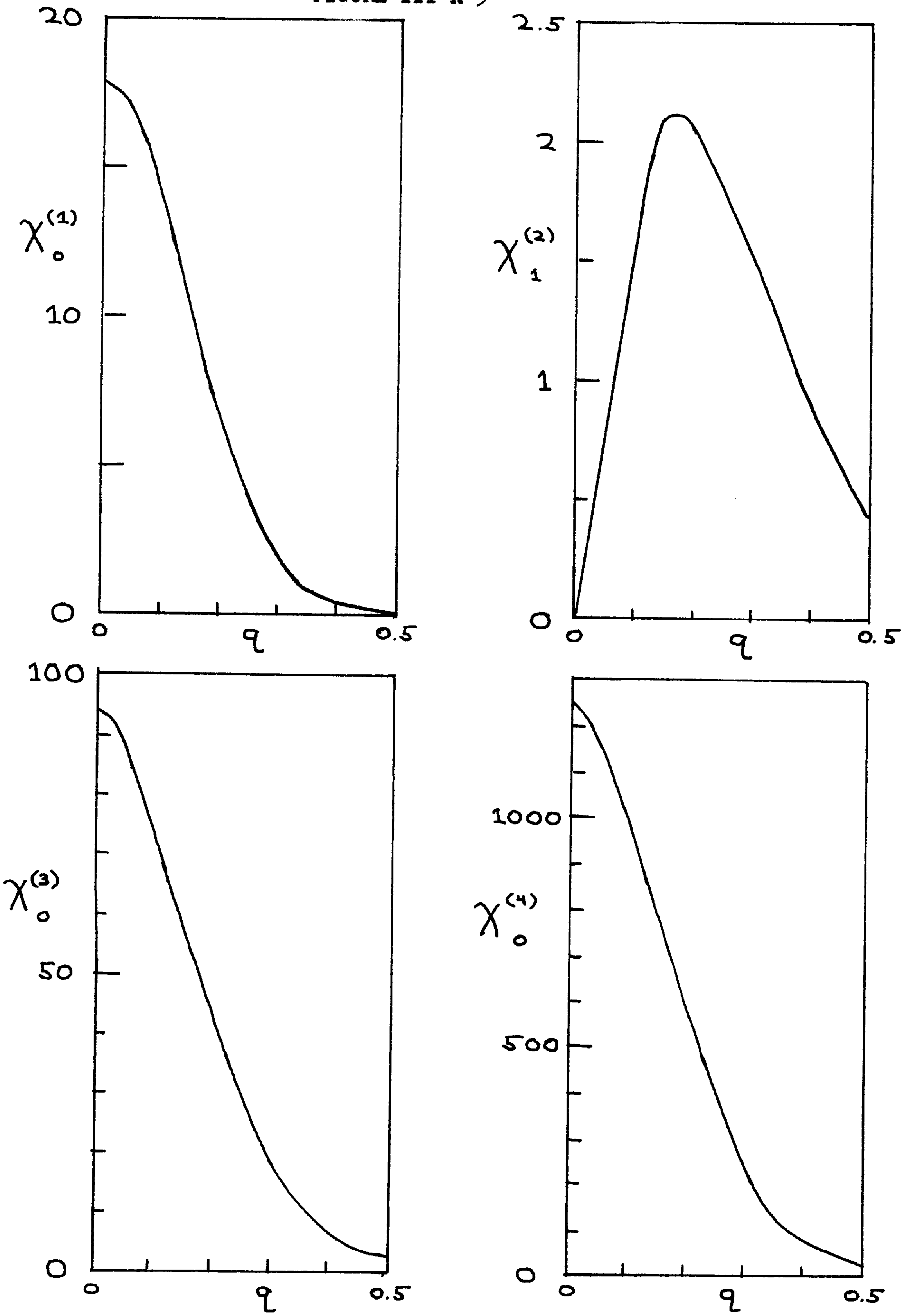
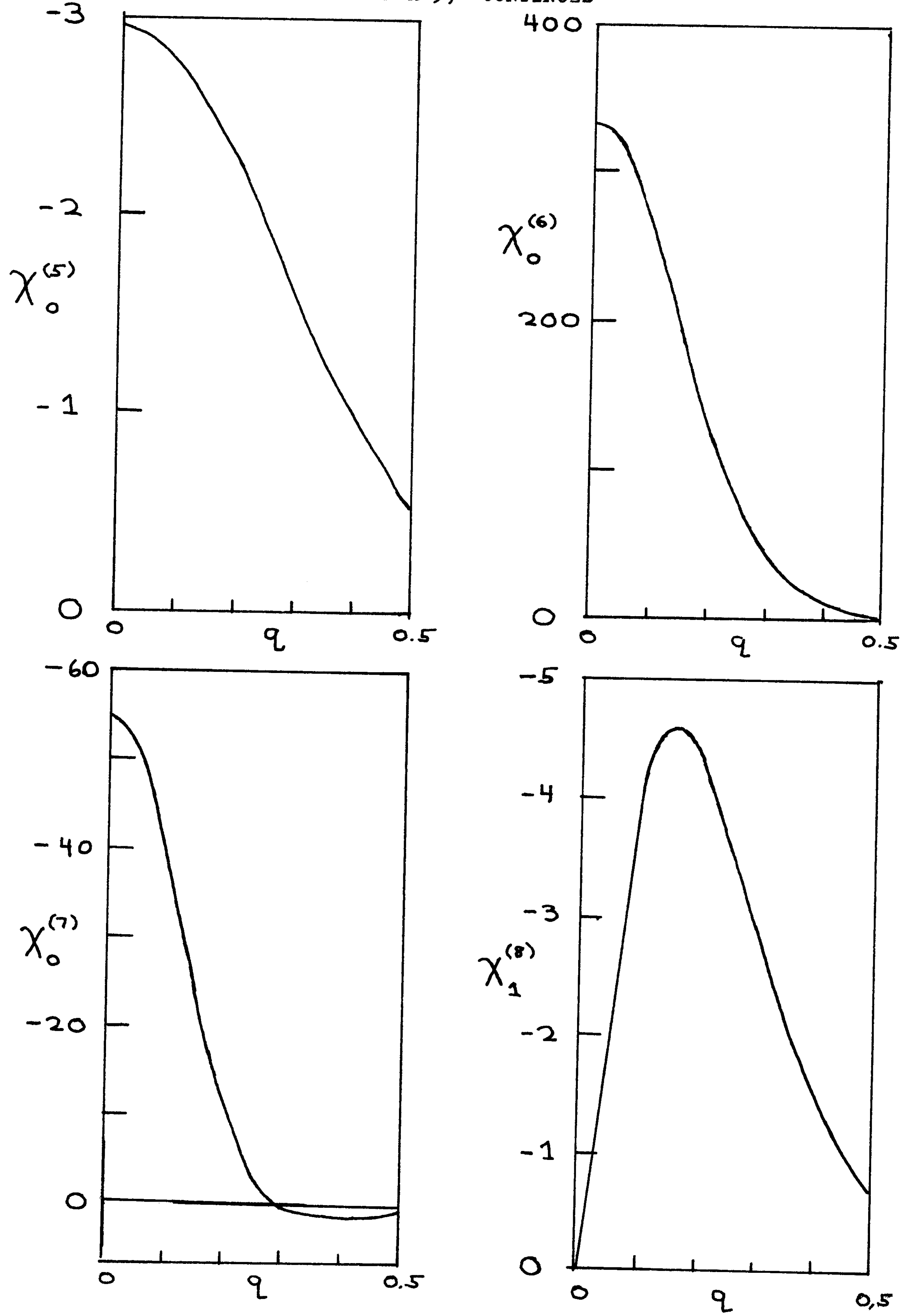


FIGURE III-A-3, CONTINUED



B. PRESENTATION OF THE DATA

In this section we will present the computer results involving the coupling constants, the bound state masses, and the decay constants f_P (f_π or f_K) and g_V (g_ρ, g_ω, or g_ϕ) which emerge from the solutions of the B-S equation (II-A-1).

Eq. (II-A-1) depends on the four parameters m (the quark mass), μ (the inverse range of the attractive interaction), Λ (the inverse range of the repulsive interaction), and M_B (the bound state mass). g^2 is treated as an eigenvalue to be found by the solution. The information about the $O(4)$ limit ($M_B \approx 0$) which we seek can be found using any very small value of M_B. The data was gathered using $M_B^2 = 10^{-4}\ m^2$. The solution of course depends only on the ratios of the three remaining masses m, μ, and Λ, so there are two free parameters to be chosen for each computer run.

As a function of the two free parameters μ/Λ and Λ/m, the computer results will be plotted in terms of the following variables:

1) Coupling constants:

$g_o^2(P)$: The value of g^2 corresponding to the $O(4)$ limit ($M_B = 0$) for the most deeply bound pseudoscalar state.

$g_o^2\ (V)$: The value of g^2 corresponding to the $O(4)$ limit for the most deeply bound vector state.

2) Quark masses:

m: The value of the p and n quark masses (in Bev) is chosen to scale all the masses so that f_P acquires its correct average value of ~ 140 Mev.

m_λ-m: By calculating the derivative $\partial M_B^2/\partial m_1$ using eq. (I-P-13), one can choose the mass difference of the λ quark so that the quantity $M_K^2 - M_\pi^2$ acquires its experimental value.

3) Predicted vector meson properties:

M_ρ: The derivative $\partial M_B^2/\partial g^2$ for both the pseudoscalar and vector mesons is calculated using eq. (I-P-18). Then a value of g^2 can be chosen so that M_π^2 acquires its experimental value. Using this value of g^2, a value of M_ρ^2 is determined. The analysis rests on the linear relation between M_B^2 and g^2, which holds in the deeply bound region.

$M_V g_V$: The computer program calculates a number for $M_V g_V$, which has the units of mass squared. Once the mass scale is fixed by f_P, this quantity can be translated into Mev^2 and compared with experiment.

$M_{K*}^2 - M_\rho^2$: Once the λ quark mass difference is determined, one can calcualte $M_{K*}^2 - M_\rho^2$ by calculating the derivative $\partial M_B^2/\partial m_1$ for the vector bound states.

Note that m, m_λ, and g^2 are all chosen to match the properties of the pseudoscalar mesons, and then predictions can be made about the vector mesons. These predicted values will all be plotted as ratios to their experimental values. (For $M_V g_V$, we have used the rough average of 0.2e Bev^2 ($e^2/4\pi$ = 1/137), or 6.06 x 10^4 Mev^2.)

Figures III-B-1, 2, and 3 show the effect of varying μ/Λ for a fixed Λ for the case of a scalar, singly regulated interaction. As can be seen, only the coupling constants depend significantly on this ratio. Using each of the three types of interactions, and using singly and doubly regulated exchange propagators, it was always found that all of the quantities except the coupling constants were insensitive to the ratio μ/Λ. For this reason, the other results will be shown only for $\mu = \frac{1}{2}\Lambda$.

Scalar Interaction: Figures III-B-4, 5, and 6 show the results for a scalar, singly regulated interaction. The values of $g_o^2(P)$ and $g_o^2(V)$ lie very near each other. For the range of variables available, the quark masses are rather light, reaching a maximum under $2\frac{1}{2}$ Bev at $m/\Lambda = 40$. (For larger values of m/Λ, the computer solution becomes rather unstable). M_ρ is too light by about a factor of two, and the quantity $M^2_{K*} - M^2_\rho$ is also low by about the same factor. $M_V g_V$ is too high, but seems to be approaching its correct value for large m/Λ.

Figures III-B-7, 8, and 9 show the results for a scalar, doubly regulated interaction. Qualitatively, the features are very similar. $g_o^2(P)$ and $g_o^2(V)$ are even closer than before. The quark masses again rise with m/Λ, but this time the quarks are heavier, ranging up to about 5 Bev. The shapes of the curves for M_ρ and $M_V g_V$ are very similar to the previous case, but each shifted downward. M_ρ is too light by a factor ranging up to four, while $M_V g_V$ varies between a value too high by a factor of $2\frac{1}{2}$ to a value too low by a factor of about two. The quantity $M^2_{K*} - M^2_\rho$ is again low by about a factor of two, and appears surprisingly insensitive to m/Λ and to the form of the propagator.

Despite the absence of a good fit to the properties of the mesons, the scalar interaction gives qualitatively good results. The lowest state is pseudoscalar, with a vector state just above it. At least for the case of a doubly regulated interaction, the quark mass can be made rather large. Although the predicted quantities do not agree with experiment, they are well within the right order of magnitude. The irreducible interaction function we have chosen is rather simple, so there is no reason to expect quantiative agreement. One can readily imagine a more complicated interaction, dominantly scalar, which would fit these properties of the mesons.

Neutral Vector Interaction: Figures III-B-10,11, and 12 show the results for a singly regulated neutral vector interaction, and Figures III-B-13, 14, and 15 show the doubly regulated case. In both cases $g_0^2(V)$ is about a factor of two above $g_0^2(P)$. For the singly regulated case, the quark masses are lighter than for the scalar interaction, rising to only 2 Bev. For the doubly regulated case, the quark masses are larger than in the corresponding scalar case, reaching 3½ Bev at $m/\Lambda = 10$. The value for $M_{K*}^2 - M_\rho^2$ is again surprisingly constant, at about 0.7 times its experimental value. The serious problem for this interaction is M_ρ. Due to the large inequality of $g_0^2(V)$ and $g_0^2(P)$, M_ρ does not become small when M_π does. By comparing the graphs, one can see that M_ρ is always roughly equal to the quark mass.

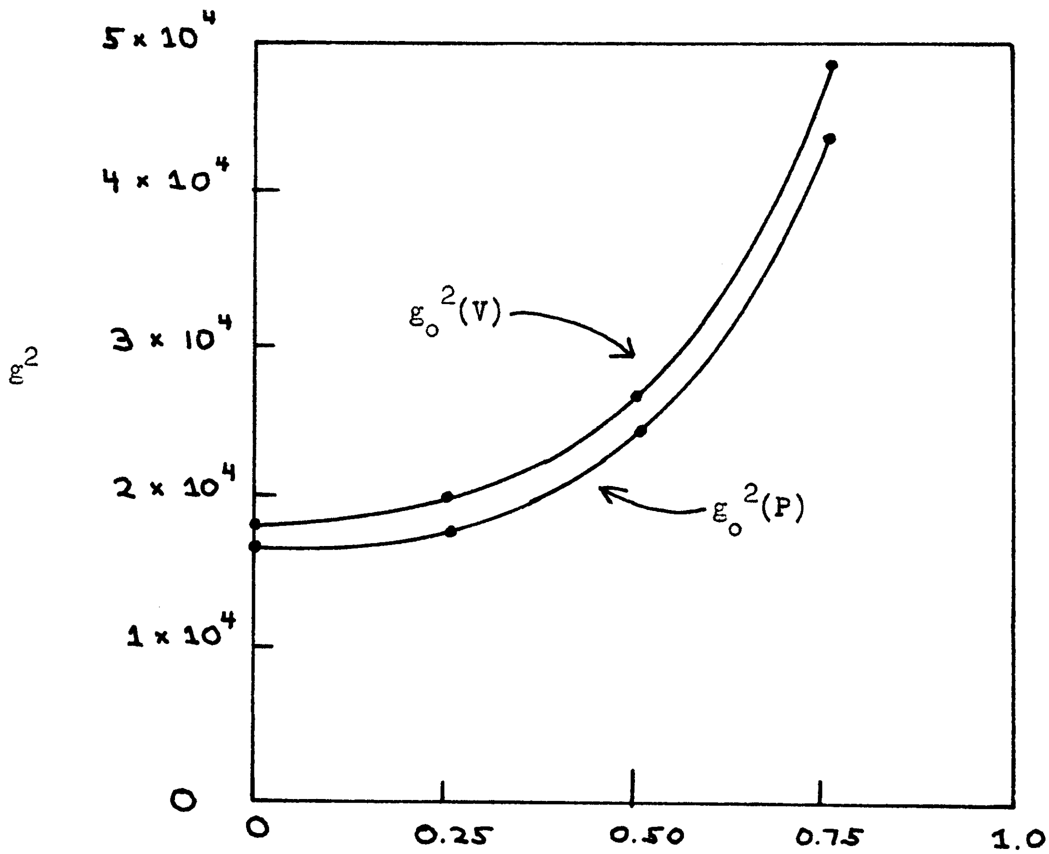
The strong spin dependence of the vector interaction is apparently a qualitative feature that is not changed by modifying the form of the interaction function $\Delta(k-q)$. Thus it appears impossible to build a quark model with a predominantly vector irreducible interaction function.

Pseudoscalar Interaction: As explained in Section II-D, the only accurate results obtained for a pseudoscalar interaction are the values of $g_o^2(V)$ and $g_o^2(P)$. These are shown in Figures III-B-16 and 17, for the cases of a singly and doubly regulated interaction, respectively. The values of $g_o^2(V)$ and $g_o^2(P)$ are extremely close, but in all cases $g_o^2(P)$ is larger. Thus the ground state is a vector, contrary to the meson spectrum.

The pseudoscalar interaction also suffers from the defect mentioned in section III-A: the lowest pseudoscalar state is part of an O(4) quartet, leading to a very inaccurate ratio of f_π/f_K. Considering these two defects, is appears that the dominant part of the irreducible interaction function cannot be pseudoscalar in character.

Numerical Accuracy: As mentioned previously, the computer solutions become unstable for $m \gtrsim 20\Lambda$. These values are shown on the graphs, but they cannot be trusted to the approximately 1% accuracy which is expected for the other points. They can probably be trusted to 5 or 10%.

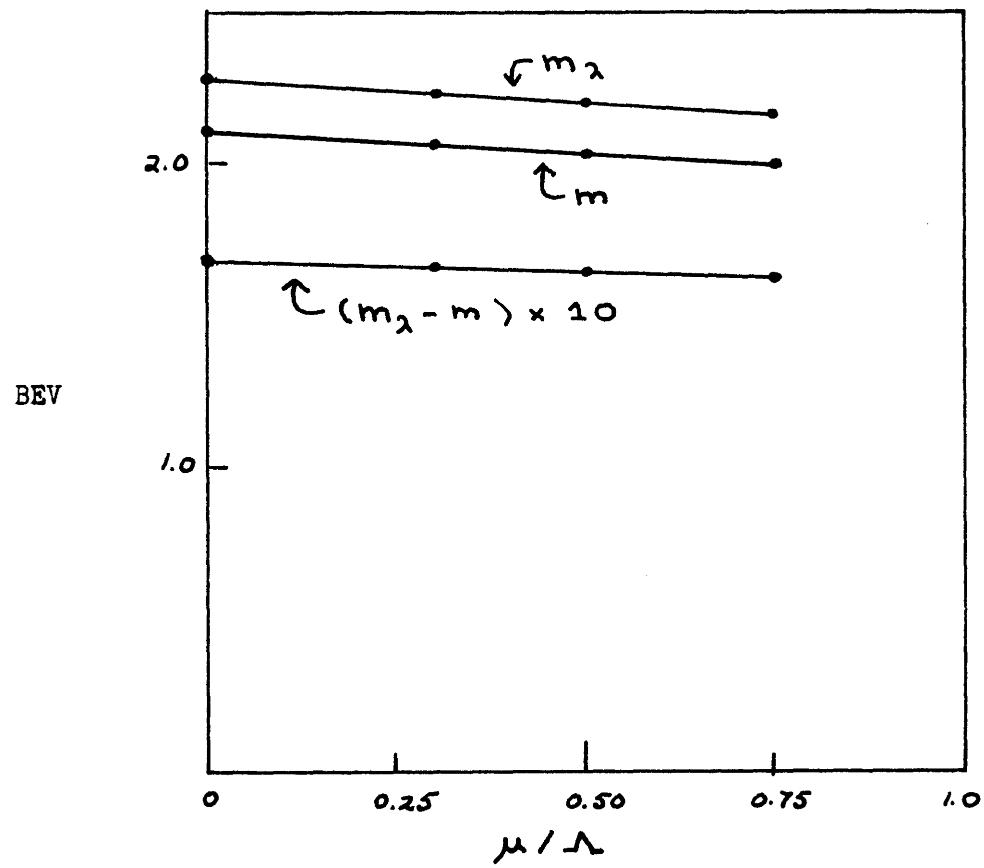
FIGURE III-B-1



Scalar, Singly Regulated Interaction

$\Lambda = 0.05$ m

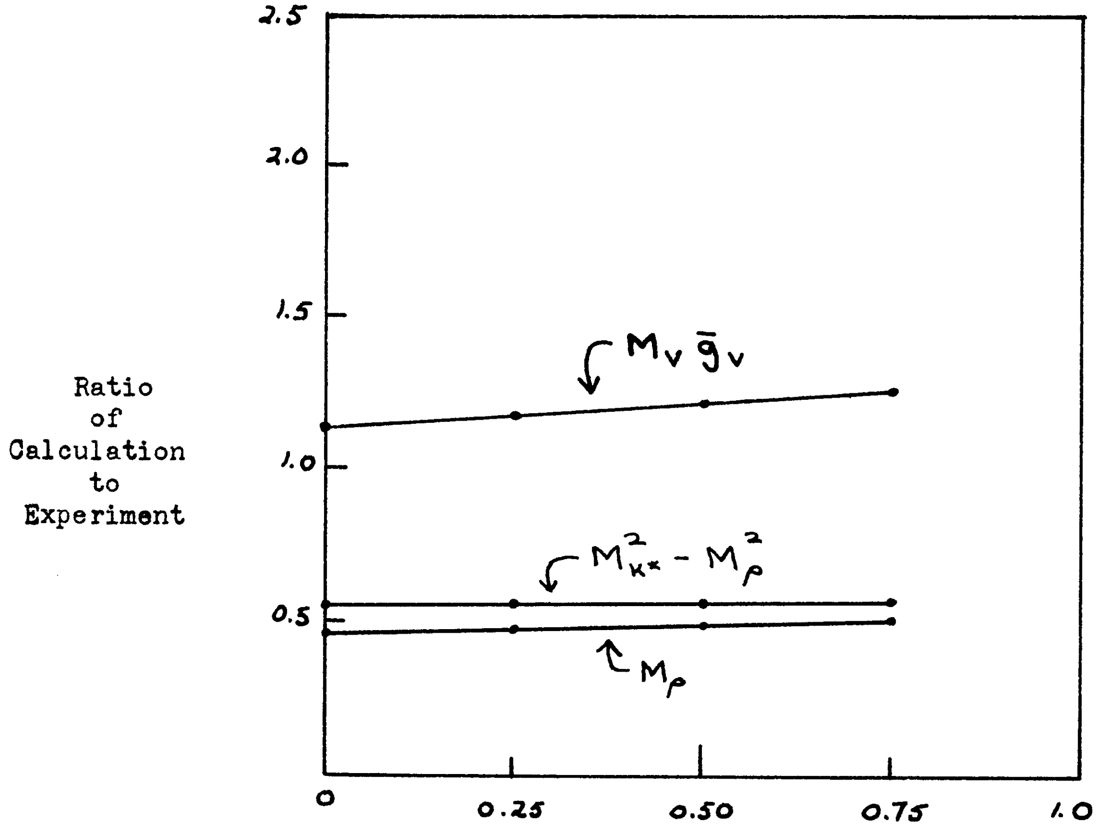
FIGURE III-B-2



Scalar, Singly Regulated Interaction

$\Lambda = 0.05$ m

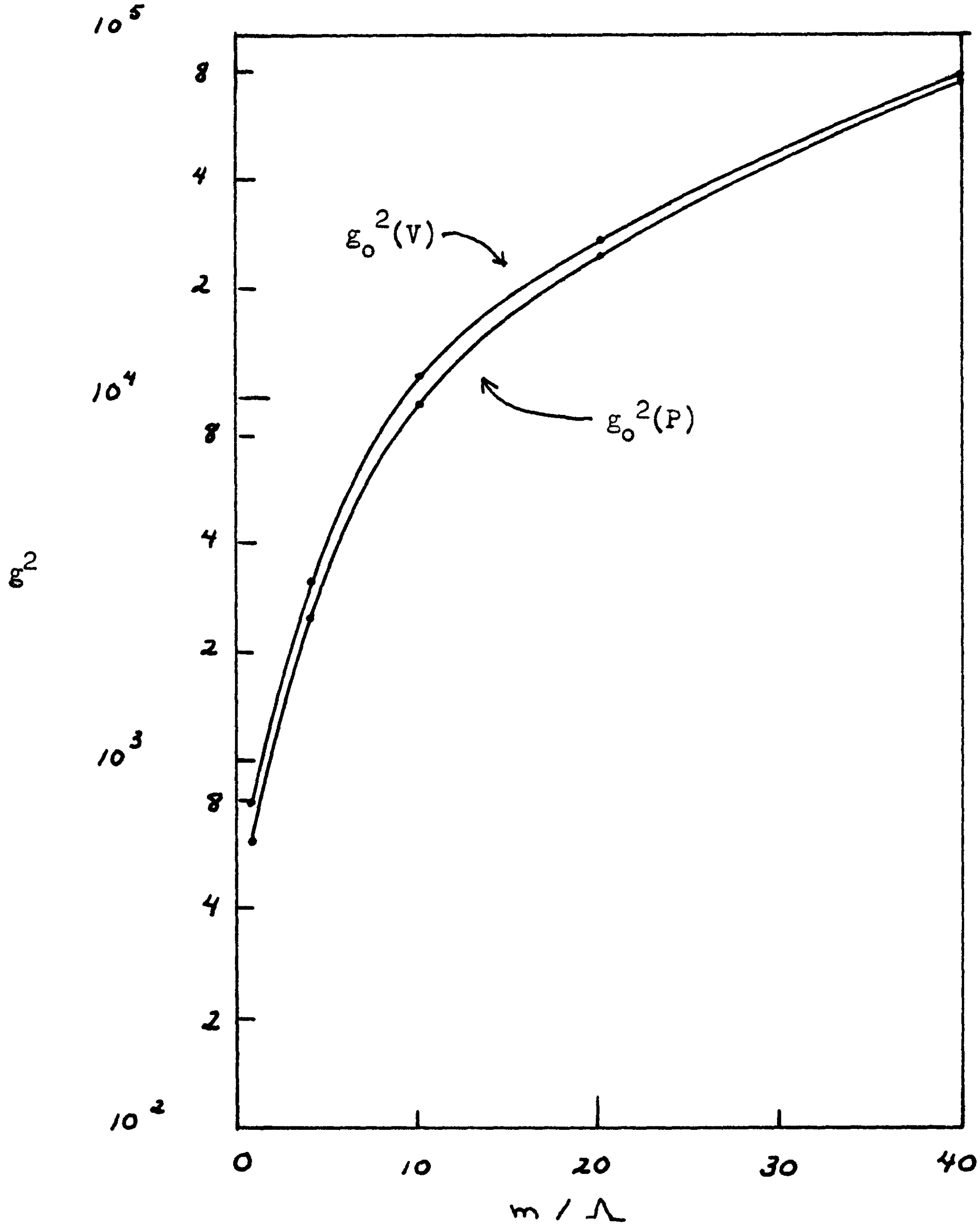
FIGURE III-B-3



Scalar, Singly Regulated Interaction

$\Lambda = 0.05$ m

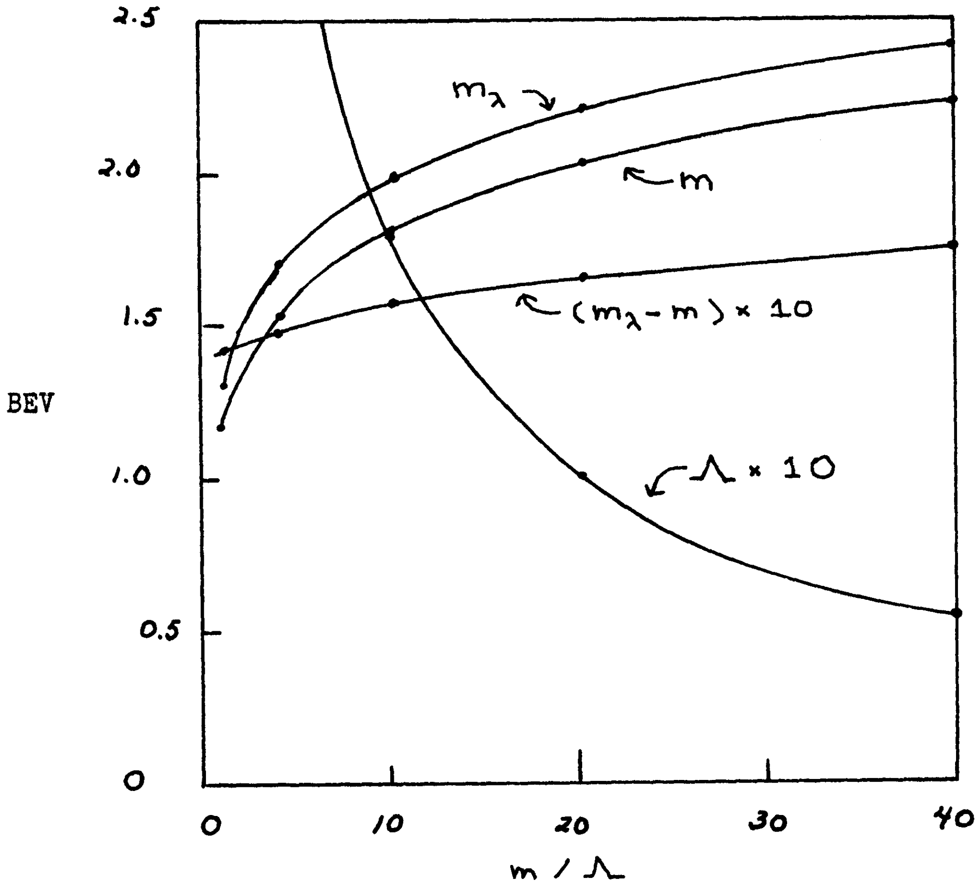
FIGURE III-B-4



Scalar, Singly Regulated Interaction

$\mu = \frac{1}{2}\Lambda$

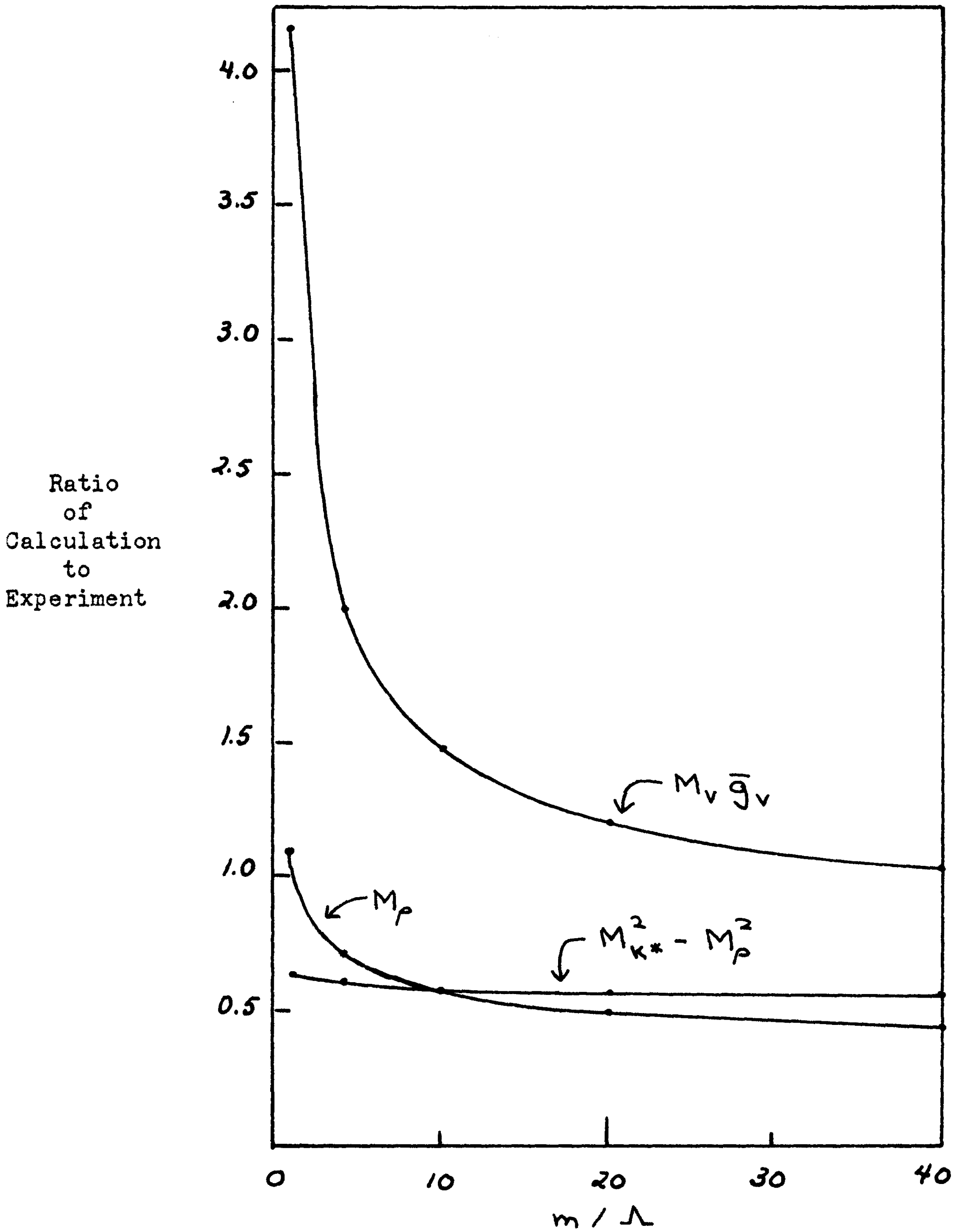
FIGURE III-B-5



Scalar, Singly Regulated Interaction

$\mu = \frac{1}{2}\Lambda$

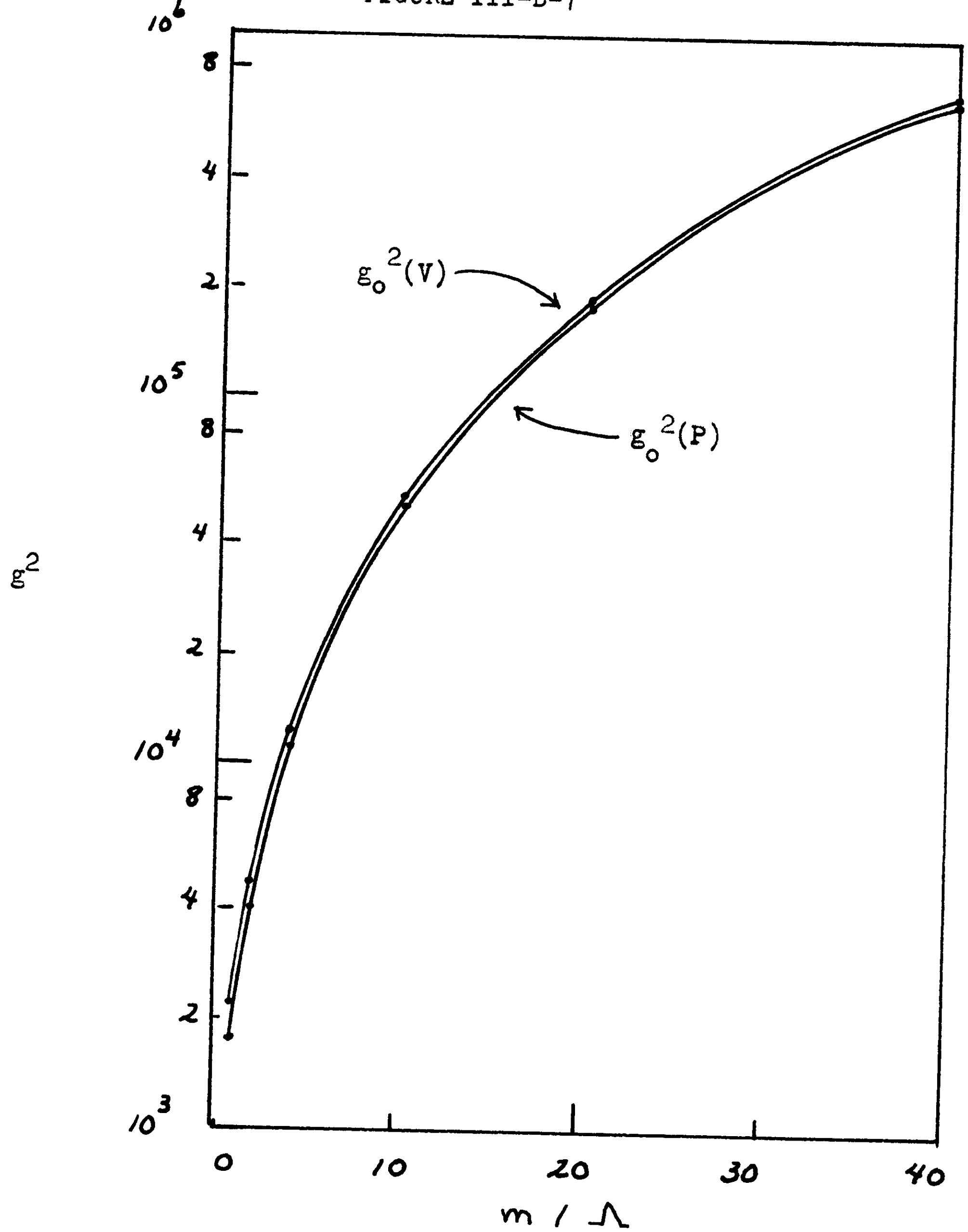
FIGURE III-B-6



Scalar, Singly Regulated Interaction

$\mu = \frac{1}{2} \Lambda$

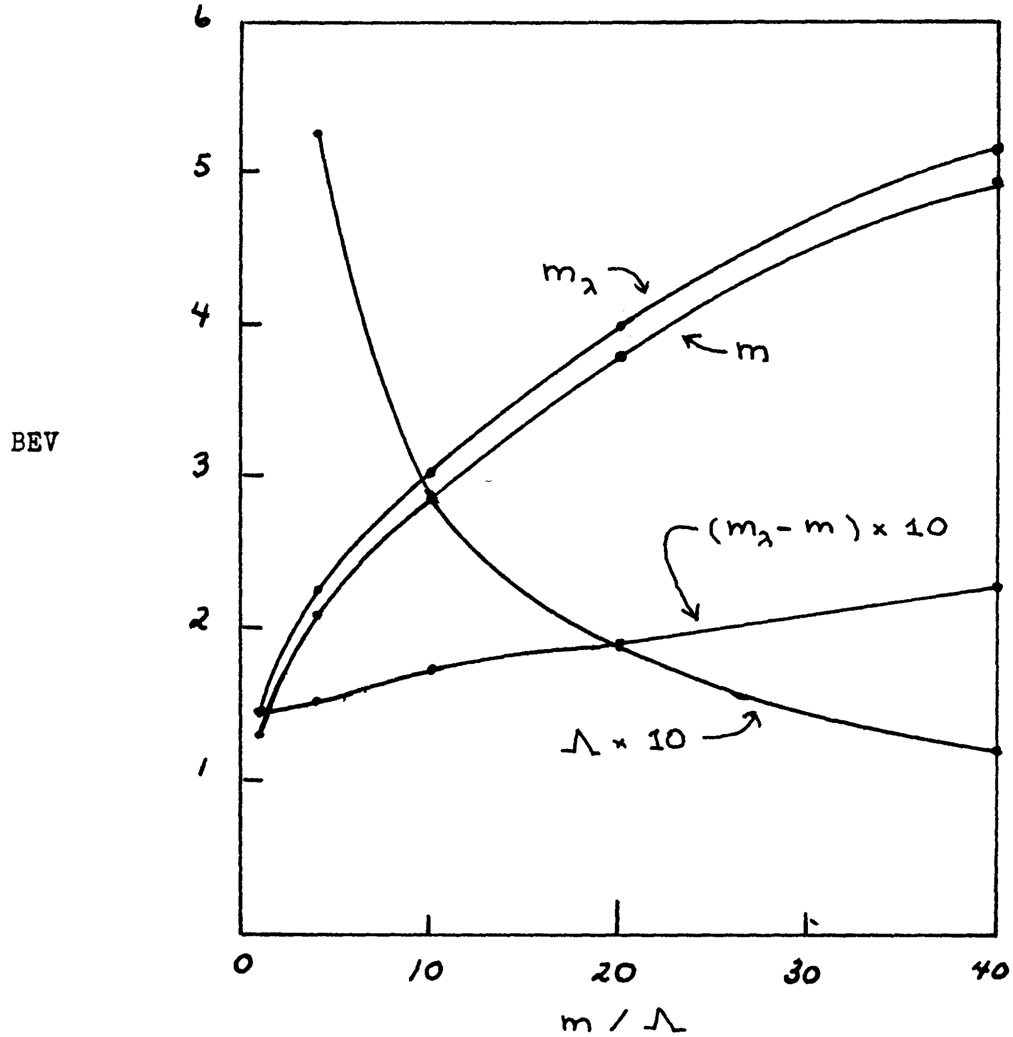
FIGURE III-B-7



Scalar, Doubly Regulated Interaction

$\mu = \frac{1}{2}\Lambda$

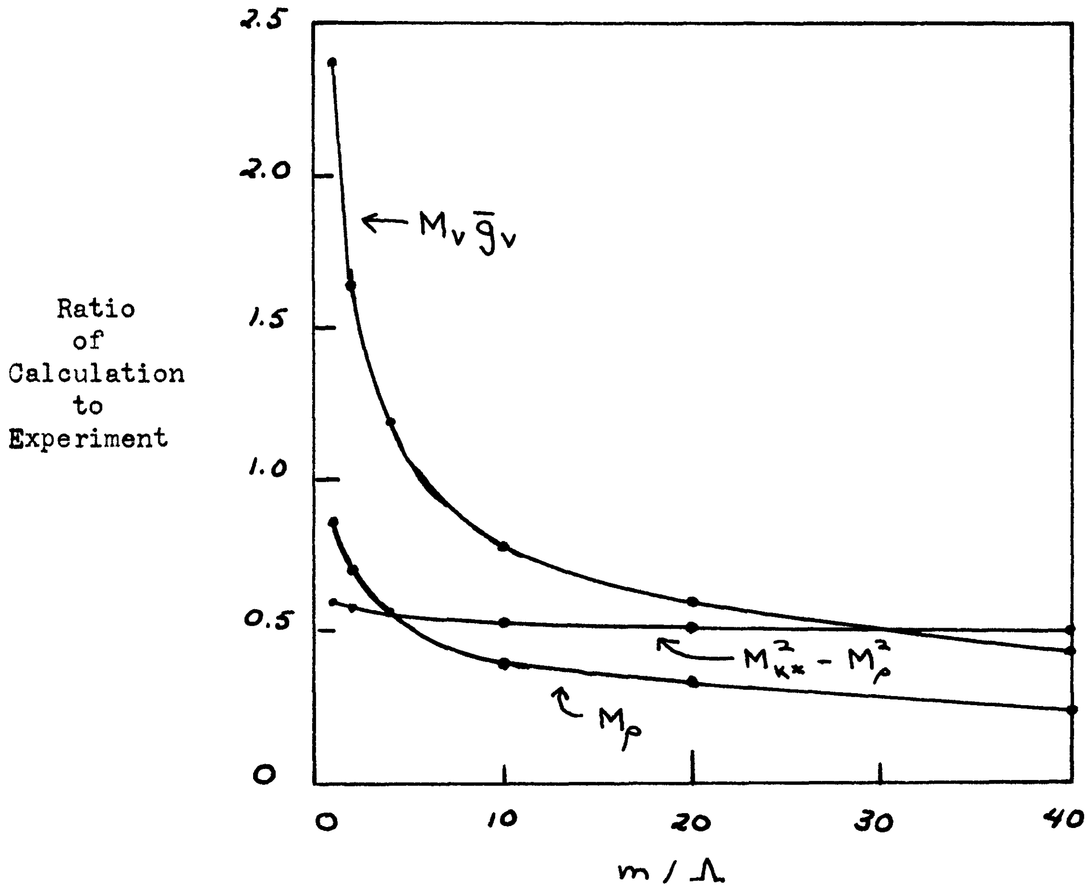
FIGURE III-B-8



Scalar, Doubly Regulated Interaction

$\mu = \frac{1}{2}\Lambda$

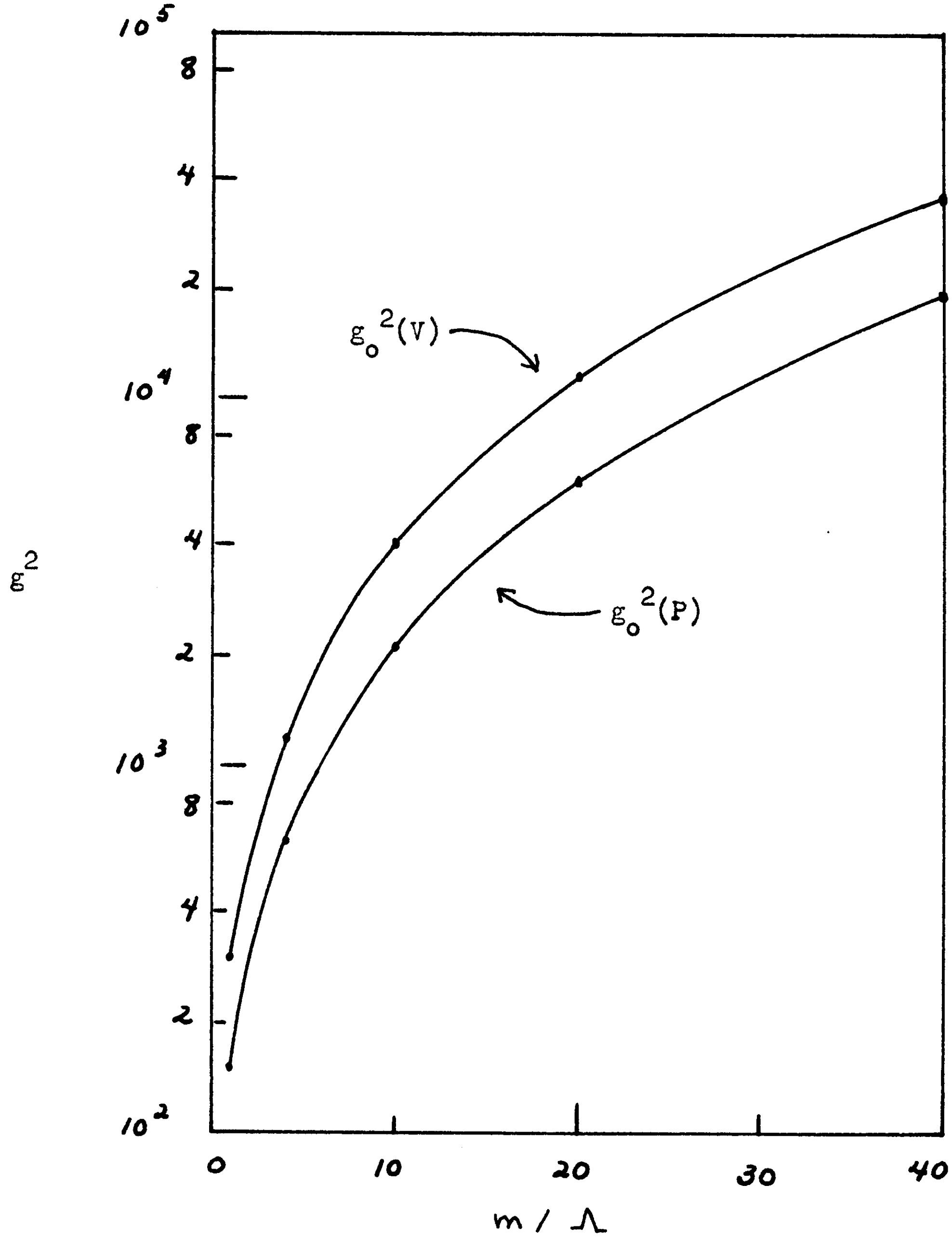
FIGURE III-B-9



Scalar, Doubly Regulated Interaction

$\mu = \frac{1}{2}\Lambda$

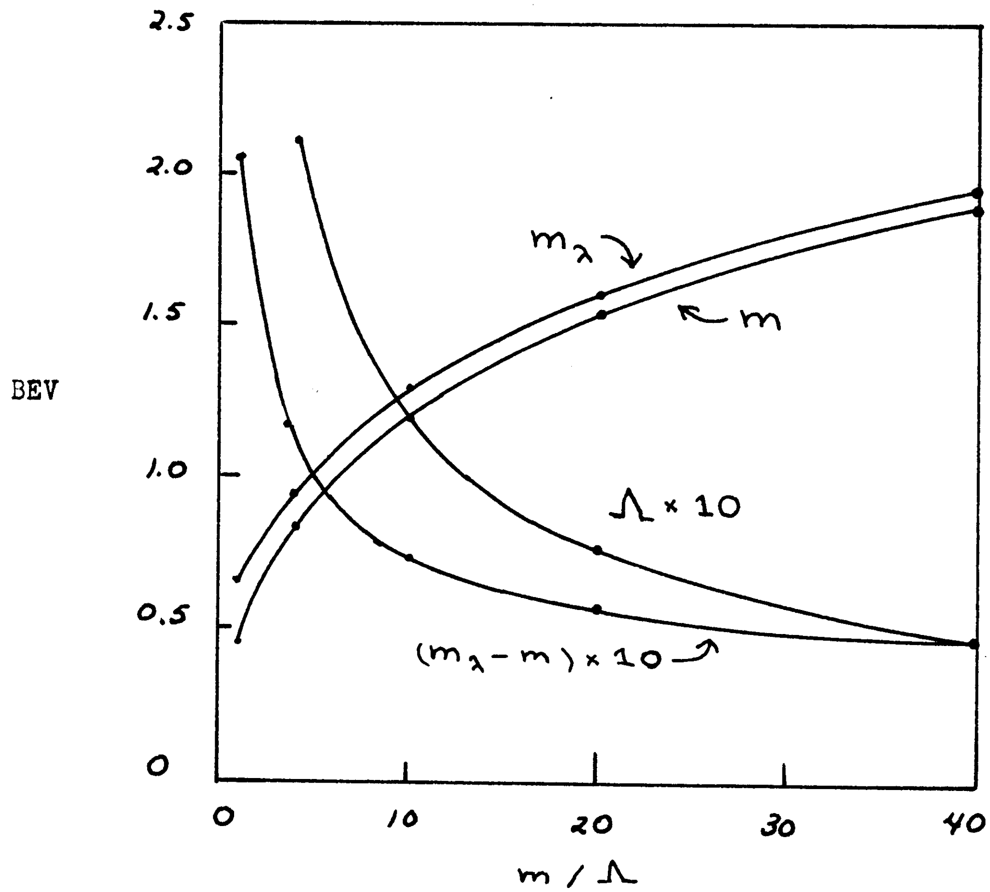
FIGURE III-B-10



Vector, Singly Regulated Interaction

$\mu = \frac{1}{2} \Lambda$

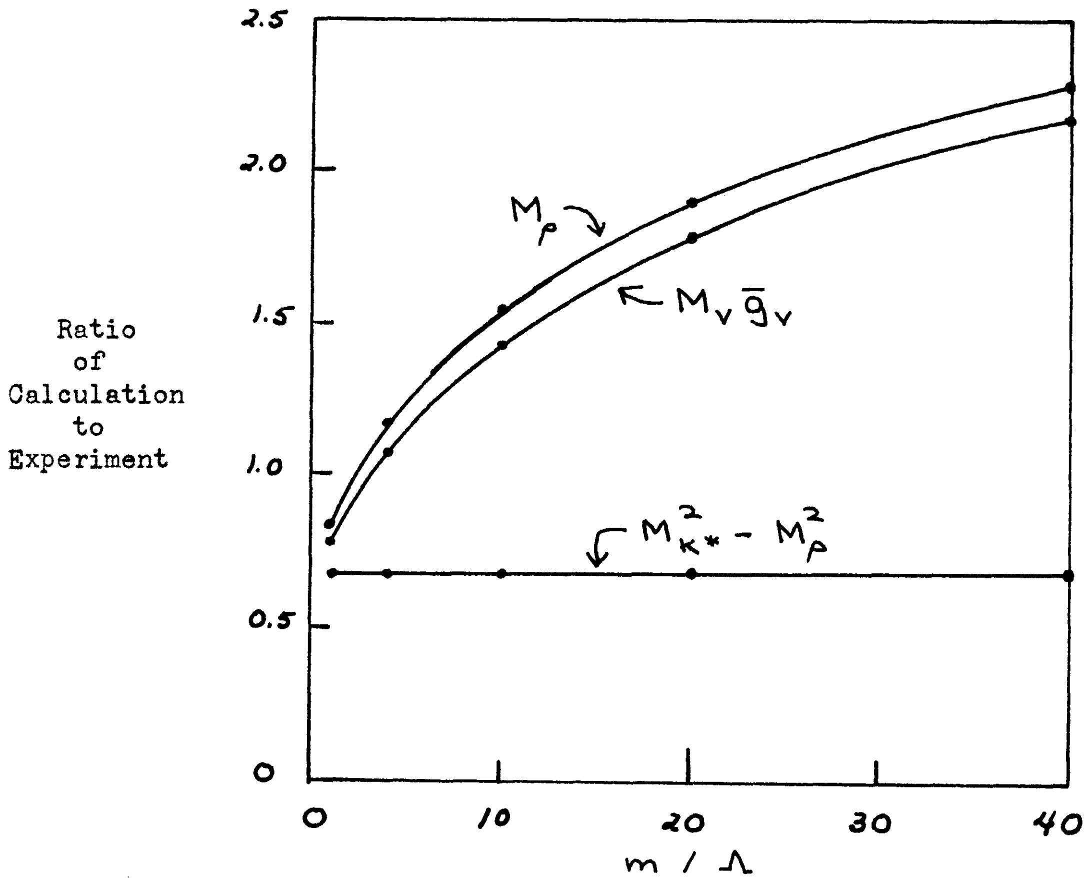
FIGURE III-B-11



Vector, Singly Regulated Interaction

$\mu = \frac{1}{2} \Lambda$

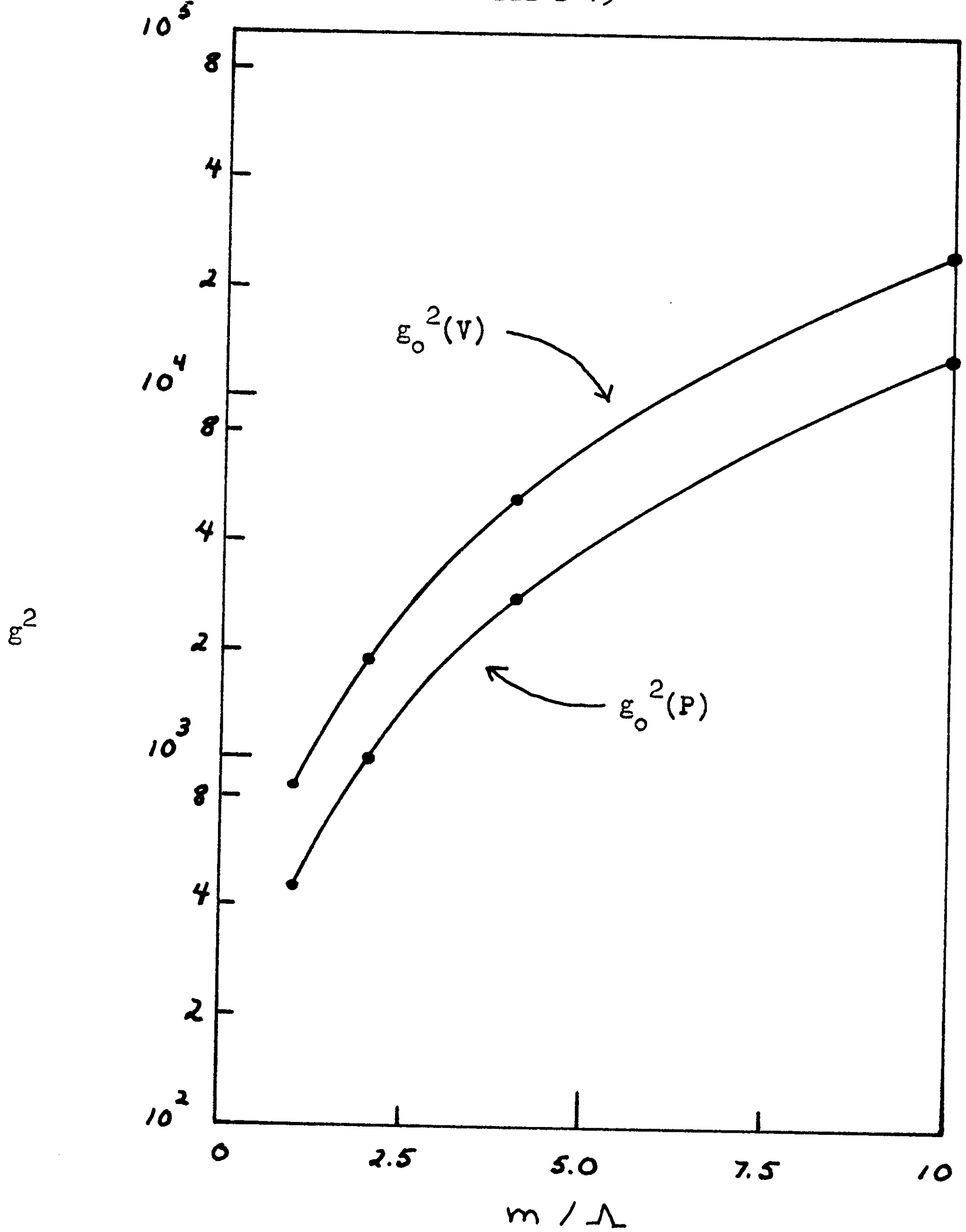
FIGURE III-B-12



Vector, Singly Regulated Interaction

$\mu = \frac{1}{2} \Lambda$

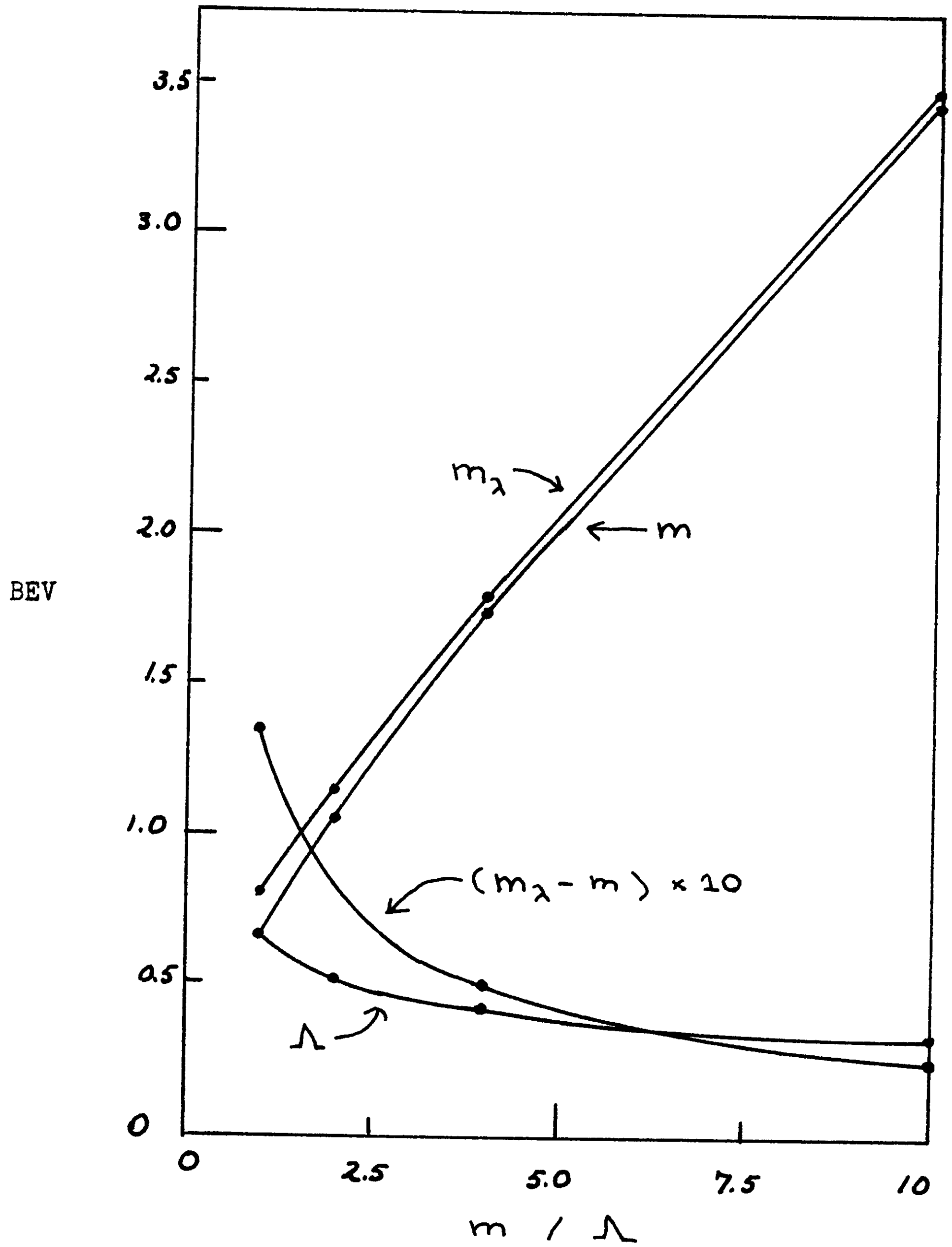
FIGURE III-B-13



Vector, Doubly Regulated Interaction

$\mu = \frac{1}{2}\Lambda$

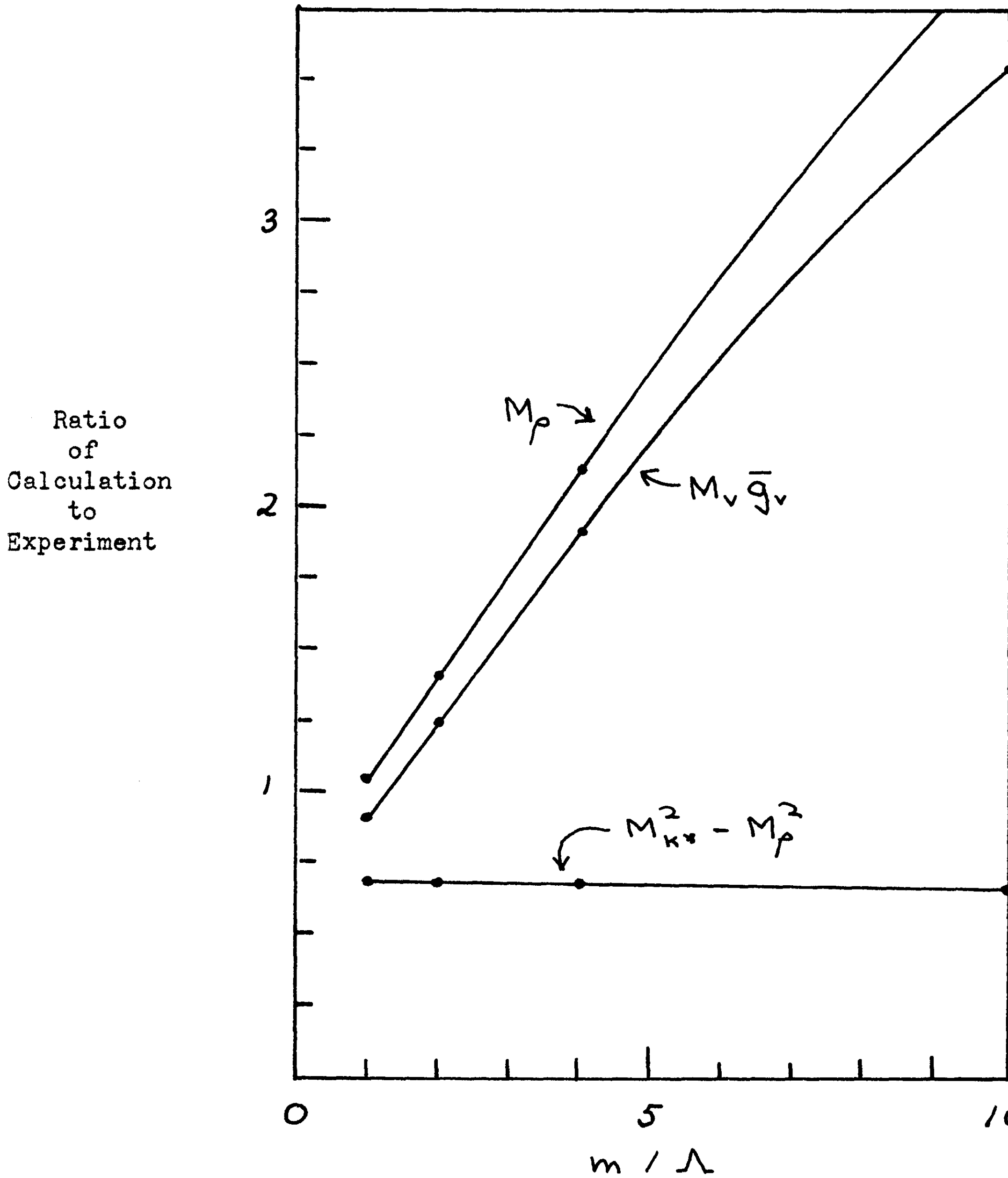
FIGURE III-B-14



Vector, Doubly Regulated Interaction

$\mu = \frac{1}{2}\Lambda$

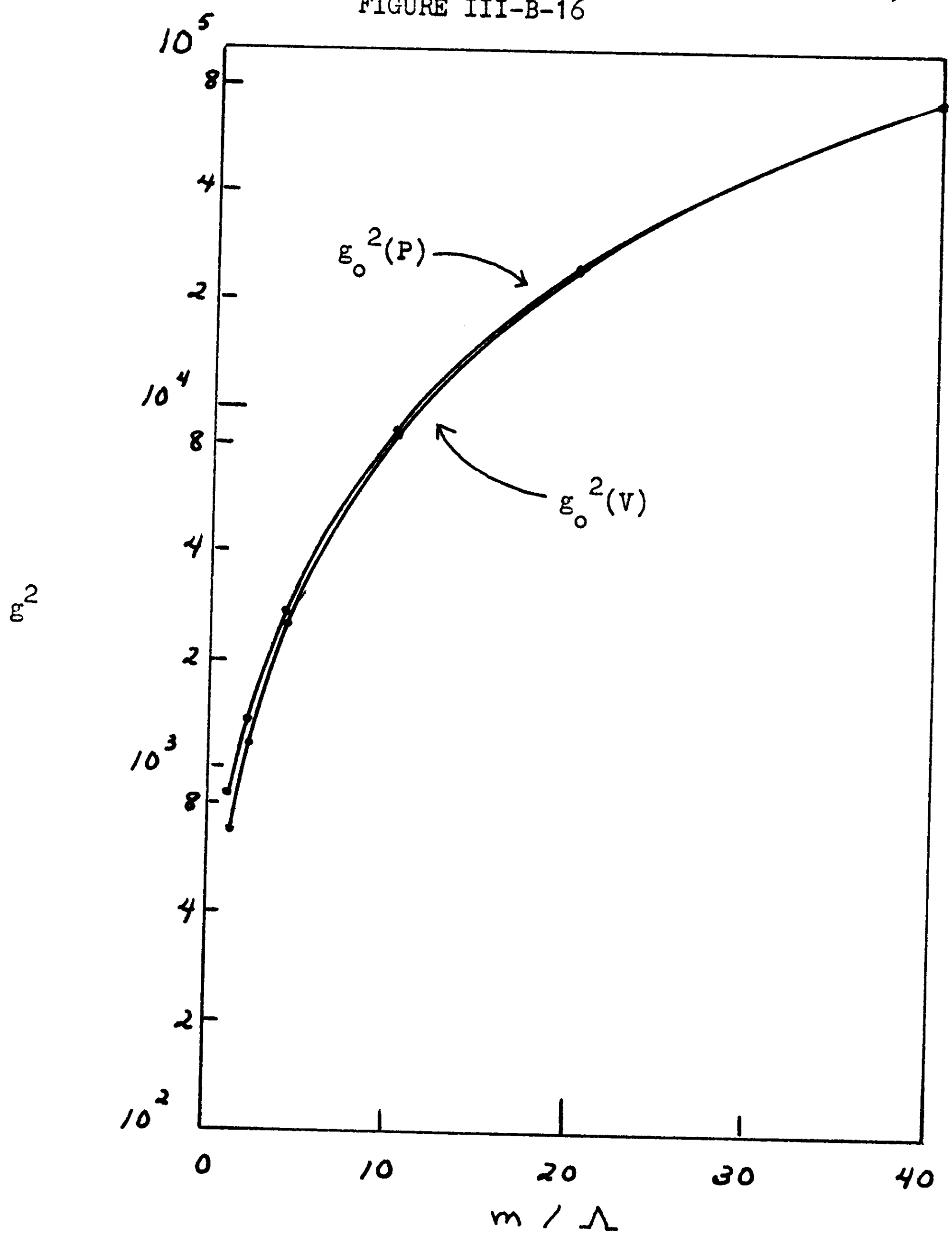
FIGURE III-B-15



Vector, Doubly Regulated Interaction

$\mu = \frac{1}{2}\Lambda$

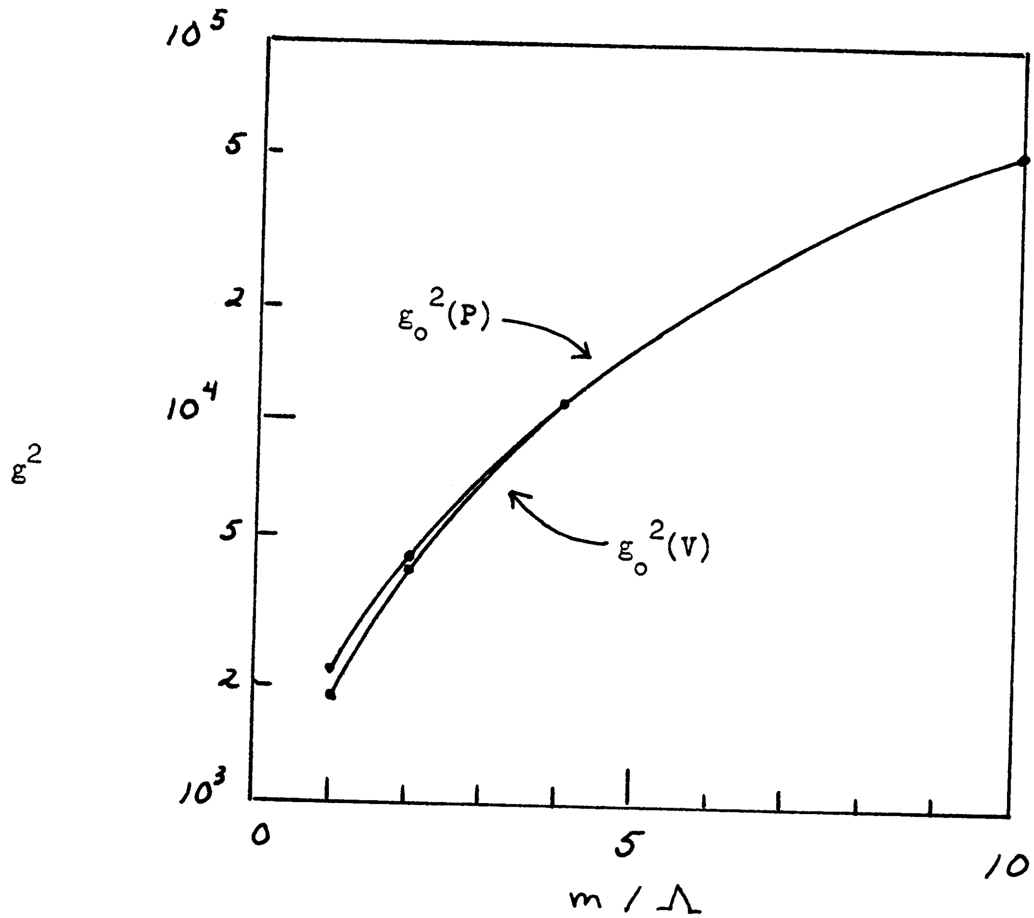
FIGURE III-B-16



Pseudoscalar, Singly Regulated Interaction

$\mu = \frac{1}{2}\Lambda$

FIGURE III-B-17



Pseudoscalar, Doubly Regulated Interaction

$\mu = \frac{1}{2} \Lambda$

C. MAGNETIC MOMENTS AND THE RADIATIVE DECAYS OF THE VECTOR MESONS

One of the most striking successes of the nonrelativistic quark model is the assumption of the additivity of quark magnetic moments. SU(3) symmetry requires each quark to have a magnetic moment proportional to its charge, so one writes a total magnetic moment operator as

$$\vec{\mu} = \mu_0 \sum_q \frac{e_q}{e} \vec{\sigma}_q , \qquad \text{(III-C-1)}$$

where μ_0 is a constant, and e_q and $\vec{\sigma}_q$ are the charge and the spin operator of the quark q. When this relation is used to calculate the magnetic moments of the baryon octet, one finds the very successful relations which were first obtained by Beg, Lee, and Pais[22] using the assumptions of SU(6). Using the value of μ_0 obtained from proton magnetic moment, one can apply the relation to the transition matrix element for $\omega \longrightarrow \pi^0 \gamma$[23]. The result is in close agreement with experiment. In this section we will test the assumption of magnetic moment additivity in our relativistic model for mesons.

The relation between the magnetic moment of the vector mesons and their B-S wavefunctions is given by eq. (I-0-17). If all the vector mesons have approximately the same wavefunction, then it can be seen that

$$\mu_V = \bar{\mu} \frac{e_V}{e} , \qquad \text{(III-C-2)}$$

i.e., each meson has a magnetic moment proportional to its charge. Thus the ratios of the magnetic moments are the ratios of the charges, which is the same as in the nonrelativistic model. This case is rather trivial, so it is not at all clear that a relativistic quark model would give the same magnetic moment ratios as the nonrelativistic model for the case of baryons.

One would like to know if the magnetic moment of a vector meson is the sum of the moments of the constituent quarks. This question can be answered most easily by assuming that the quarks have only an anomalous magnetic moment, so that Γ_3 in eq. (I-0-17) can be written as

$$e\Gamma_3 = \mu_A \, \sigma_{3\nu} \, (P_{2\nu} - P_{1\nu}) , \tag{III-C-3}$$

where $e = e_1 - e_2$ is the total charge of the meson, and μ_A is the total magnetic moment of the quarks. Then the equation becomes, after Wick rotation,

$$\begin{aligned} \mu = \frac{\mu_A}{2M} \int \frac{d^4\bar{q}}{(2\pi)^4} \, \mathrm{Tr} \Big\{ & \bar{\chi}(\bar{p},\bar{q},e_2)\, \sigma_{13} \\ & \times \chi(\bar{p},\bar{q},e_1)\, i\, [\gamma\cdot(i\bar{q}+\bar{p})+m] \Big\} , \end{aligned} \tag{III-C-4}$$

where the polarization vectors are given by

$$\begin{aligned} e_1 &= (0,0,1,0) \\ e_2 &= (1,0,0,0) \end{aligned} \tag{III-C-5}$$

This expression has been calculated for a number of cases, and it was always found that μ was larger than μ_A, by factors ranging from 8 to 70. The results are listed in Table III-C-1.

One would also like to know if the matrix element for $V \longrightarrow P\,\gamma$ is related simply to the quark magnetic moments. The formalism for these radiative decays is developed in section I-N. The amplitude is expressed in terms of a constant $\bar{\beta}$ which is related to the B-S wavefunction by eq. (I-N-12). To simplify the problem, we will again assume that the quarks have only an anomalous magnetic moment. We will also calculate the answer only in the limit of $M_P \longrightarrow M_V$, which means that $t \longrightarrow 0$ and $k \longrightarrow 0$ in eq. (I-N-12). This latter approximation is clearly unrealistic, but it is necessary in order to allow $\bar{\beta}$ to be expressed in terms of Wick rotated wavefunctions. We can still ask the following nontrivial question: in a fictitious world in which $M_\pi \approx M_\omega$, would the decay rate $\omega \longrightarrow \pi^\circ \gamma$ be related to the quark magnetic moments as in the nonrelativistic model?

Using the anomalous moment, Γ_2 in eq. (I-N-12) can be written as

$$e\Gamma_2 = \mu_A \, \sigma_{2\nu} \, (P_{2\nu} - P_{1\nu}) . \tag{III-C-6}$$

In the limit $M_P \longrightarrow M_V$, the equation becomes after Wick rotation

$$\begin{aligned} \bar{\beta} = i \frac{\mu_A}{M} \int \frac{d^4\bar{q}}{(2\pi)^4} \, \mathrm{Tr} \{ \, & \bar{\chi}^{(P)} (\bar{p}, \bar{q}) \, (\sigma_{21} + i \sigma_{24}) \\ & \chi^{(V)} (\bar{p}, \bar{q}, e^{(V)}) \; i \, [\gamma \cdot (i\bar{q} + \bar{p}) + m] \} \end{aligned} \tag{III-C-7}$$

where

$$M = M_V = M_P$$

and

$$\bar{p} = (0, 0, 0, \tfrac{1}{2} M) .$$

By using parity invariance as expressed in eq. (I-E-13), one can show that the contribution of the σ_{24} term vanishes.

The nonrelativistic amplitude for $V \rightarrow P\gamma$ corresponds to the assumption $|\bar{\beta}| = 2\mu_A$. Table III-C-1 shows the values that have been found for $|\beta| / 2\mu_A$. It can be seen that these values are near unity, ranging from .98 to 1.25. Given the previous result, this result is somewhat surprising.

We have found that the magnetic moments of the vectors mesons are not given by additivity of the quark moments, but the matrix elements for $V \rightarrow P\gamma$ can be calculated by additivity with reasonable accuracy. The conclusion, however, is that we have found no simple relation between hadronic magnetic moments and the transition matrix elements for $V \rightarrow P\gamma$. Thus, the success of the nonrelativistic quark model prediction remains either a mystery or a coincidence.

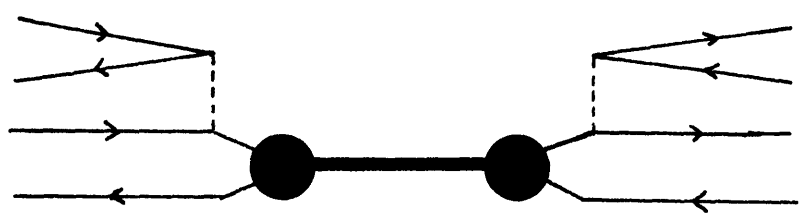
TABLE III-C-1

VALUES OF μ AND $\bar{\beta}$

μ/Λ	$\acute{m}/\Lambda$	μ/μ_A	$\lvert\bar{\beta}\rvert\ /2\mu_A$
Scalar, Doubly Regulated Interaction:			
$\frac{1}{2}$	1	29.8	1.01
$\frac{1}{2}$	4	13.9	1.16
$\frac{1}{2}$	10	8.01	1.25
0	4	12.0	1.19
$\frac{1}{4}$	4	12.9	1.18
$\frac{3}{4}$	4	14.9	1.15
Scalar, Singly Regulated Interaction:			
$\frac{1}{2}$	10	16.2	1.14
Neutral Vector, Doubly Regulated Interaction:			
$\frac{1}{2}$	2	70.4	.988
$\frac{1}{2}$	10	60.7	.975

D. DAUGHTERS OF THE VECTOR MESONS

All of the vector meson states which were found belong to O(4) quartets, or four-vector representations. The fourth member of the quartet is then a meson with $J^{PC} = 0^{+-}$, which is sometimes referred to as the daughter of the vector meson. States with this J^{PC} cannot be constructed from a quark-antiquark pair in the non-relativistic quark model. In the relativistic model, the corresponding statement is that the B-S wavefunction vanishes identically when both quark and antiquark are on mass shell. Thus, there are no poles corresponding to these states in the quark-antiquark S-matrix, but there are poles in the off mass shell Green's function. There will be poles in the two quark two antiquark S-matrix, as can be seen for example in the following diagram:



Thus, these states correspond to nonrelativistic states of two quarks and two antiquarks.

In the O(4) limit these states are degenerate with the vector mesons, which means they have zero mass when $g^2 = g_o^{\,2}(V)$. As g^2 varies slightly from $g_o^{\,2}(V)$, $M_B^{\,2}$ varies linearly, with a slope given by $\partial M_B^{\,2}/\partial g^2$. The same happens with the ρ mass, except the slope is different. Thus,

$$\frac{M_{scalar}}{M_\rho} = \sqrt{\frac{\left(\frac{\partial M_B^2}{\partial g^2}\right)_{scalar}}{\left(\frac{\partial M_B^2}{\partial g^2}\right)_{vector}}} \quad . \tag{III-D-1}$$

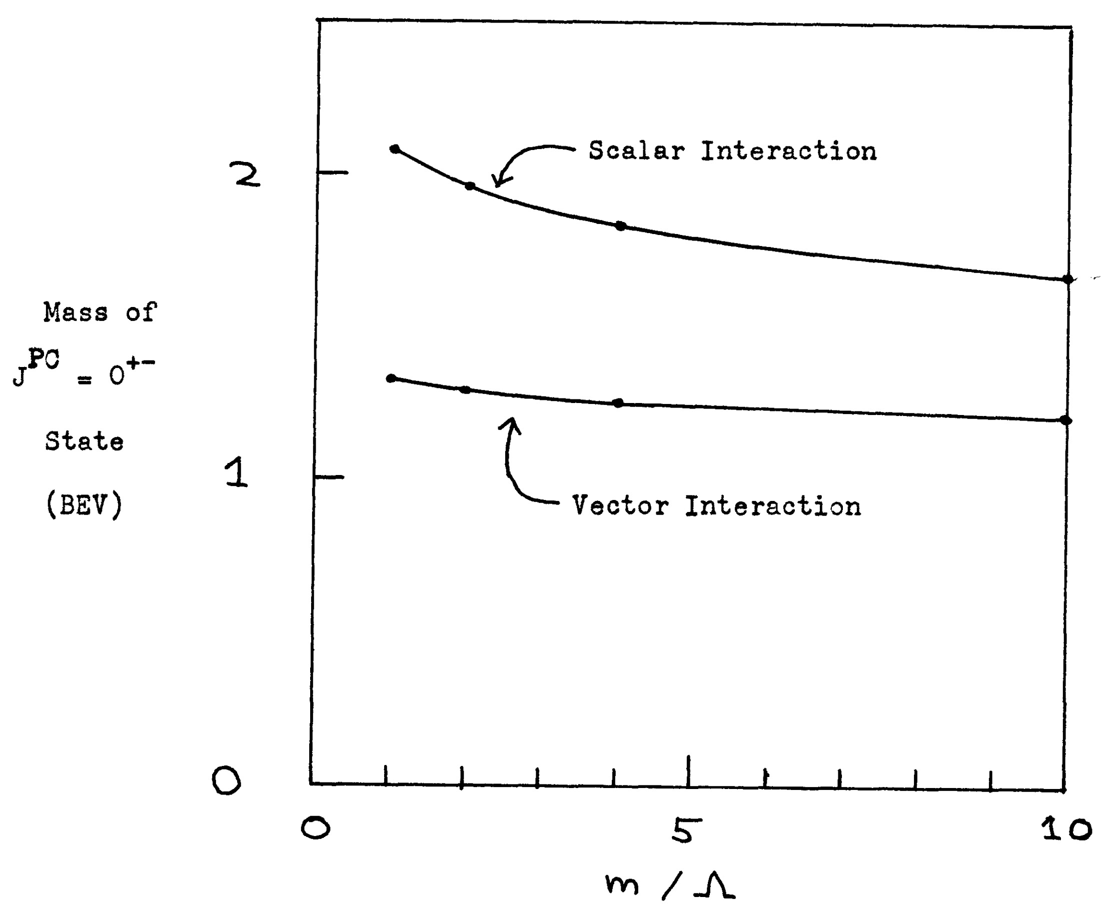
The resulting masses for the scalar mesons (computed using the experimental value of M) are shown in Figure III-D-1. The mass is in all cases large. For a scalar, doubly regulated interaction the mass varies from about 1½ to 2 Bev, and for a neutral vector, doubly regulated interaction it stays at about 1¼ Bev.

The anomalous feature of these states is that they are ghosts. That is, when the wavefunctions are used in the normalization integral of eq. (I-J-13), the integration gives a result with the wrong sign. By reviewing the derivation of the normalization condition, one sees that these ghost states correspond to poles in the Green's function with residues of the opposite sign from that expected from intermediate positive norm states.

It is unclear how these ghost states should be interpreted, but one can be thankful that they are so massive. These masses are very likely outside the range of validity of the O(4) limit calculations, which means that we do not really know if the states are ghosts or not.

However, it is clear that the form of the B-S equation we are using does yield ghost solutions for $g^2 \lesssim g_0^2(V)$. Since the equation is not unitary, the presence of these ghost states should not be very surprising. But their presence does indicate that a unitary theory would have significantly different behavior near the O(4) limit. Thus, one acquires a certain degree of doubt about the validity of the nonunitary B-S equation in the deeply bound region.

FIGURE III-D-1



Doubly Regulated Interactions

$\mu = \frac{1}{2}\Lambda$

CONCLUSION

The main goal of the research has been to find out if the fully relativistic formalism of the Bethe-Salpeter equation could produce deeply bound states which are nonetheless nonrelativistic in character. To this question our answer is negative. By using an appropriate phenomenological interaction function (one which falls off as $1/q^6$ or faster), one can obtain deeply bound solutions with quark momenta which are small compared to the quark mass. However, these states still have spinor structures which are highly relativistic. In particular, we have always found that $\chi_{++} \approx \pm \chi_{--}$. We have also found that these solutions do not have simple quark additivity properties, as in the nonrelativistic model. In particular, the magnetic moment of a vector meson is not equal to the sum of the quark magnetic moments.

A subsidiary goal has been to learn if such a deeply bound relativistic model can successfully account for the properties of the mesons. We have not answered this question, but we have discovered some grounds for optimism. It is found that the relativistic model justifies the successful quadratic form of the Gell-Mann-Okubo mass formula, and that it predicts the correct ratio of f_π / f_K (in contrast to the nonrelativistic model). Using a very simple interaction function, one can predict meson properties to within factors of about 4. It is conceivable that the data could be fit by a more complicated interaction function, which would be dominantly but not necessarily entirely scalar. To continue, one should also consider mesons of higher spin.

Finally, we point out that the Bethe-Salpeter equation, using the regulated single particle exchange interaction function, has several undesirable features. The most important is the fact that it yields $J^{PC} = 0^{+-}$ ghost solutions, as discussed in section III-D. A second undesirable feature is the ease with which tachyons are produced. As g^2 is increased, $M_B{}^2$ moves freely through zero and becomes negative. It is hard to believe that the interaction just happens to be strong enough to produce a pion at $M_{\pi}^2 = .02$ Bev^2, but is not strong enough to produce tachyons. Thus, it would appear preferable to have an equation with an internal mechanism which avoids tachyons.

APPENDIX A

NOTATION

Coordinates:

$$x_\mu \equiv (x_1, x_2, x_3, x_4) \equiv (x, y, z, it)$$
$$\equiv (\vec{x}, it)$$

$$x_4 \equiv i x_0 \,, \qquad x_0 \equiv t$$

Momenta:

$$q_\mu \equiv (q_1, q_2, q_3, q_4) \equiv (q_x, q_y, q_z, iE_q)$$
$$\equiv (\vec{q}, iE_q)$$

$$q_4 \equiv i q_0 \,, \qquad q_0 \equiv E_q$$

$$\int d^4 k \equiv \int dk_0\, dk_1 dk_2\, dk_3$$

$$k \cdot q \equiv \sum_{\mu=1}^{4} k_\mu q_\mu \equiv \vec{k}\cdot\vec{q} - E_k E_q$$

Wick rotation:

When a momentum is Wick rotated so that all four components q_1, q_2, q_3, q_4 are real, then it is indicated by a bar: $\overline{q}$,$\overline{k}$, etc. Then

$$\int d^4\bar{k} \equiv \int dk_1\, dk_2\, dk_3\, dk_4$$

The symbols q and k are used to represent the Euclidean lengths of such vectors. The symbols $\hat{q}$ and $\hat{k}$ indicate unit vectors in the same direction. (We use also the specialized notation $\bar{p}_\mu \equiv -ip_\mu$, where $2p_\mu$ is the momentum of the bound state. Note that $\bar{p}_\mu$ is real in the rest frame.)

Four-Vector Operators:

$J_i^{\dagger}$	Hermitean adjoint of J_i, $i = 1,2,3$.
$J_4^{\dagger}$	- (Hermitean adjoint of J_4).

Physical State Normalization:

$$\langle P' \mid P \rangle \equiv 2P_0\, \delta^3(\vec{P}' - \vec{P}).$$

Dirac Matrices (Pauli Notation):

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\gamma_i \equiv \begin{pmatrix} 0 & -i\sigma_i \\ i\sigma_i & 0 \end{pmatrix} \qquad \gamma_4 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_{\mu\nu} \equiv \frac{i}{2}[\gamma_\mu, \gamma_\nu]$$

$$\gamma_5 \equiv \gamma_1\gamma_2\gamma_3\gamma_4 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$$

Totally antisymmetric tensor:

$$\epsilon_{\mu\nu\lambda\sigma}; \qquad \epsilon_{1234} \equiv 1.$$

APPENDIX B

O(4) EIGENFUNCTIONS

The Gegenbauer polynomials are discussed in many mathematical references[19], so here we will simply list the important properties.

The Gegenbauer polynomials are denoted by $C_n^{\alpha}(x)$, where n is a non-negative integer. They may be defined by the generating relation

$$(1 - 2xz + z^2)^{-\alpha} = \sum_{n=0}^{\infty} C_n^{\alpha}(x) z^n . \tag{A-B-1}$$

They satisfy the differential equation

$$\frac{d}{dx} (1-x^2) \frac{dC_n^{\alpha}(x)}{dx} - (2\alpha - 1) x \frac{dC_n^{\alpha}(x)}{dx} + n(n+2\alpha) C_n^{\alpha}(x) = 0, \tag{A-B-2}$$

and they can be expressed in polynomial form as

$$C_n^{\alpha}(x) = \sum_{m=0}^{[\frac{n}{2}]} (-1)^m \frac{(n-m+\alpha-1)!}{(n-2m)!\, m!\, (\alpha-1)!} (2x)^{n-2m} . \tag{A-B-3}$$

[x] denotes the largest integer less then or equal to x. They obey the following useful recursion relations:

$$\frac{dC_n^{\alpha}(x)}{dx} = 2\alpha\, C_{n-1}^{\alpha+1}(x), \tag{A-B-4}$$

$$2\alpha x\, C_n^{\alpha+1}(x) = (n+1)\, C_{n+1}^{\alpha}(x) + 2\alpha\, C_{n-1}^{\alpha+1}(x), \tag{A-B-5}$$

$$2x\, C_n^{\alpha}(x) = C_{n+1}^{\alpha}(x) + C_{n-1}^{\alpha}(x) - C_{n+1}^{\alpha-1}(x), \tag{A-B-6}$$

$$\begin{aligned} 2(n+\alpha)\, x\, C_n^{\alpha}(x) &= (n+1)\, C_{n+1}^{\alpha}(x) \\ &\quad + (n+2\alpha-1)\, C_{n-1}^{\alpha}(x), \end{aligned} \tag{A-B-7}$$

and

$$(n+\alpha)\, C_n^{\alpha}(x) = \alpha\left[C_n^{\alpha+1}(x) - C_{n-2}^{\alpha+1}(x)\right]. \tag{A-B-8}$$

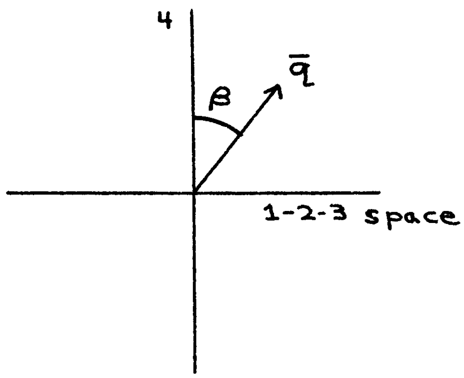
They obey the orthonormality condition

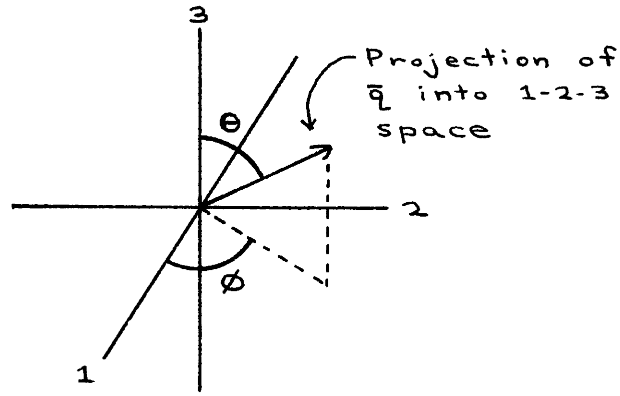
$$\begin{aligned} &\int_{-1}^{1} dx\, (1-x^2)^{\alpha-\frac{1}{2}}\, C_m^{\alpha}(x)\, C_n^{\alpha}(x) \\ &\qquad = \frac{\pi\, (n+2\alpha-1)!}{2^{2\alpha-1}\, (n+\alpha)\, n!\, [(\alpha-1)!]^2}\, \delta_{mn} \end{aligned} \tag{A-B-9}$$

To discuss four dimensional spherical harmonics, one must define four dimensional polar coordinates. If $\bar{q}$ is a Euclidean four-vector, define the polar angles β, θ, and ϕ by

$$\begin{aligned} \bar{q}_1 &= q \sin\beta \sin\theta \sin\phi \\ \bar{q}_2 &= q \sin\beta \sin\theta \cos\phi \\ \bar{q}_3 &= q \sin\beta \cos\theta \\ \bar{q}_4 &= q \cos\beta\,; \end{aligned} \tag{A-B-10}$$

The coordinates are shown in the following diagrams:





The four dimensional spherical harmonics are then given by

$$Y_{n\ell m}(\beta, \theta, \phi) = \mathcal{C}^{\ell}_{n}(\beta)\, \mathcal{Y}_{\ell m}(\theta, \phi), \tag{A-B-11}$$

where $\mathcal{Y}_{\ell m}(\theta, \phi)$ is the usual three dimensional spherical harmonic function, and

$$\mathcal{C}^{\ell}_{n}(\beta) \equiv \left[\frac{2^{2\ell+1}\,(n+1)\,(n-\ell)!}{\pi\,(n+\ell-1)!}\right]^{1/2} \times \ell!\, \sin^{\ell}\beta\; C^{\ell+1}_{n-\ell}(\cos\beta). \tag{A-B-12}$$

The indices obey the inequalities

$$0 \le \ell \le n$$
$$\text{and} \qquad |m| \le \ell. \tag{A-B-13}$$

The orthonormality relation is

$$\int d\Omega \, Y^*_{n'\ell'm'}(\Omega) \, Y_{n\ell m}(\Omega) = \delta_{n'n} \, \delta_{\ell'\ell} \, \delta_{m'm} \tag{A-B-14}$$

where

$$d\Omega = \sin^2\beta \, d\beta \, \sin\theta \, d\theta \, d\phi . \tag{A-B-15}$$

The $\mathcal{C}^{\alpha}_{n}(\beta)$ satisfy the orthonormality relation

$$\int_0^{\pi} \sin^2\beta \, d\beta \, \mathcal{C}^{\ell}_{n'}(\beta) \, \mathcal{C}^{\ell}_{n}(\beta) = \delta_{n'n} . \tag{A-B-16}$$

The Feynman propagator, once Wick rotated to the Euclidean momentum region, can be expanded in four dimensional spherical harmonics according to the following formula[13]:

$$\frac{1}{(\bar{k}-\bar{q})^2 + \mu^2} = \frac{8\pi^2}{(r+s)^2} \sum_{n\ell m} \frac{1}{n+1} \left(\frac{r-s}{r+s}\right)^n \times Y^*_{n\ell m}(\hat{k}) \, Y_{n\ell m}(\hat{q}) , \tag{A-B-17}$$

where

$$\begin{aligned} r &= \sqrt{(k+q)^2 + \mu^2} \\ s &= \sqrt{(k-q)^2 + \mu^2} , \end{aligned} \tag{A-B-18}$$

with k and q representing the Euclidean length of the vectors, and $\hat{a}_\mu = a_\mu/a$.

In section II-C, it was found convenient to use the properties of the O(4) eigenfunctions expressed in terms of traceless symmetric tensors. Here we will outline the justification of those properties.

Given a Euclidean four-vector k_μ, there is an n'th rank traceless symmetric tensor which can be formed from it and $\delta_{\mu\nu}$, unique up to a normalization constant. This tensor will be denoted by curly brackets:

$$\begin{aligned} \{k_{\mu_1} k_{\mu_2} \dots k_{\mu_n}\} \equiv\ & k_{\mu_1} k_{\mu_2} \dots k_{\mu_n} \\ & + \text{constant} \times k^2 \left[\delta_{\mu_1\mu_2} k_{\mu_3} \dots k_{\mu_n} + \text{symmetric terms} \right] \\ & + (\text{terms involving two or more } \delta_{\mu\nu}\text{'s}). \end{aligned} \qquad \text{(A-B-19)}$$

The normalization is fixed by the first term. The first four such tensors are listed below:

$$\begin{aligned} \{1\} &= 1 \\ \{k_\mu\} &= k_\mu \\ \{k_\mu k_\nu\} &= k_\mu k_\nu - \tfrac{1}{4} k^2 \delta_{\mu\nu} \\ \{k_\mu k_\nu k_\lambda\} &= k_\mu k_\nu k_\lambda - \tfrac{1}{6} k^2 \left[\delta_{\mu\nu} k_\lambda + \delta_{\mu\lambda} k_\nu + \right. \\ & \qquad \left. + \delta_{\nu\lambda} k_\mu \right] \end{aligned} \qquad \text{(A-B-20)}$$

These tensors obey the recursion relation

$$\begin{aligned} \{k_{\mu_1} \dots k_{\mu_n}\} =\ & \frac{1}{n} \sum_{i=1}^{n} k_{\mu_i} \{ k_{\mu_1} \dots k_{\mu_i} \dots k_{\mu_n} \} \\ & - \frac{1}{2n(n-1)} \sum_{i=1}^{n} \sum_{j=i+1}^{n} \delta_{\mu_i\mu_j} k^2 \{ k_{\mu_1} \dots k_{\mu_i} \dots k_{\mu_j} \dots k_{\mu_n} \}, \end{aligned} \qquad \text{(A-B-21)}$$

where the arrowed circles indicate that the enclosed factor is deleted. Rather than prove this relation directly at this point, we will adopt it, in addition to the first two of eqs. (A-B-20), as the definition of the curly brackets. It is clear that the curly brackets so defined are symmetric and have the leading term given in eq. (A-B-19), but it remains to be shown that they are traceless.

The next step is to prove the relation

$$k_\nu \{k_{\mu_1} \dots k_{\mu_{n-1}} k_\nu\} = \frac{n+1}{2n} k^2 \{k_{\mu_1} \dots k_{\mu_{n-1}}\} . \qquad \text{(A-B-22)}$$

which can be shown using mathematical induction and the recursive definition of eq. (A-B-21). Using the above relation and the recursive definition, one can express the trace $\{k_\mu k_\mu k_{\mu_3} \dots k_{\mu_n}\}$ in terms of the traces of lower rank curly bracket tensors. Then by mathematical induction, it can be shown that all these traces vanish.

It will be necessary to know how to multiply $\{k_{\mu_1} \dots k_{\mu_n}\}$ by k_μ, but it is easier to first ask how to exchange the indices of a vector outside the curly brackets and a vector inside the curly brackets. It can be shown, again by induction, that

$$\begin{aligned} & k_\lambda \{k_{\mu_1} \dots k_{\mu_{n-1}} k_\sigma\} - k_\sigma \{k_{\mu_1} \dots k_{\mu_{n-1}} k_\lambda\} \\ & = \frac{k^2}{2n} \sum_{i=1}^{n-1} \delta_{\lambda \mu_i} \{k_{\mu_1} \dots k_{\mu_i} \dots k_{\mu_{n-1}} k_\sigma\} \qquad \text{(A-B-23)} \\ & \qquad - (\lambda \leftrightarrow \sigma) . \end{aligned}$$

This relation can be used to modify the first term on the right hand side of the recursive definition, so that the same index always appears on the outside of the curly bracket. The result is

$$
\begin{aligned}
k_\nu \{ k_{\mu_1} \cdots k_{\mu_n} \} &= \{ k_\nu k_{\mu_1} \cdots k_{\mu_n} \} \\
&+ \frac{k^2}{2(n+1)} \sum_{i=1}^{n} \delta_{\nu\mu_i} \{ k_{\mu_1} \cdots k_{\mu_i} \cdots k_{\mu_n} \} \\
&- \frac{k^2}{2n(n+1)} \sum_{i=1}^{n} \sum_{j=i+1}^{n} \delta_{\mu_i\mu_j} \{ k_\nu k_{\mu_1} \cdots k_{\mu_i} \cdots k_{\mu_j} \\
&\qquad \cdots k_{\mu_n} \} .
\end{aligned}
\tag{A-B-24}
$$

It is now time to consider the relation between the traceless symmetry tensors and the O(4) eigenfunctions discussed earlier. In general there exists a traceless symmetric tensor $e_{\mu_1 \ldots \mu_n}(n \ell m)$ such that

$$
Y_{n\ell m}(\hat{k}) = \{ \hat{k}_{\mu_1} \cdots \hat{k}_{\mu_n} \} e_{\mu_1 \ldots \mu_n}(n \ell m) . \tag{A-B-25}
$$

We will want some special cases of the relation.

It is fairly obvious that

$$
Y_{n00}(\hat{k}) \propto \{ \hat{k}_4 \cdots \hat{k}_4 \}^{(n)} , \tag{A-B-26}
$$

where the superscript (n) indicates the rank of the tensor. In terms of the Gegenbauer polynomials, this becomes

$$
C^1_n(\cos\beta) \propto \{ \hat{k}_4 \cdots \hat{k}_4 \}^{(n)} . \tag{A-B-27}
$$

To fix the constant of proportionally, use eq. (A-B-3) to express the leading term (highest power of x) of $C^{\alpha}_{n}(x)$:

$$C^{\alpha}_{n}(x) = \frac{2^{n}\,(n+\alpha-1)!}{n!\,(\alpha-1)!}\,x^{n} + \cdots \,. \tag{A-B-28}$$

The leading term of the right hand side of eq. (A-B-27) is $\cos^{n}\beta$, so the proportionality constant is determined:

$$\{\hat{k}_{4}\ldots\hat{k}_{4}\} = \frac{1}{2^{n}}\,C^{1}_{n}(\cos\beta)\,. \tag{A-B-29}$$

The above argument is not a complete proof, but the above equation can be rigorously verified by mathematical induction. Use the recursive definition of the curly brackets, and the recursion relation (A-B-7) for the Gegenbauer polynomials.

Similarly, it is obvious that

$$Y_{n1m}(\hat{k}) \propto \{\hat{k}_{i}\,\hat{k}_{4}\ldots\hat{k}_{4}\}^{(n)}\,e_{i}(m)\,, \tag{A-B-30}$$

where $e_i(m)$ is the O(3) spin one polarization vector. (The indices i, j, k, and ℓ will take on values of 1, 2, or 3.) By matching the leading terms as above, one arrives at

$$\{\hat{k}_{i}\,\hat{k}_{4}\ldots\hat{k}_{4}\}^{(n)} = \frac{1}{2^{n-1}\,n}\,C^{2}_{n-1}(\cos\beta)\,\hat{k}_{i}\,. \tag{A-B-31}$$

The inductive proof of the above equation relies on relation (A-B-29), and the Gegenbauer polynomial recursion relation (A-B-6).

For $\ell = 2$, one starts with

$$Y_{n2m}(\hat{k}) \propto \{\hat{k}_i \hat{k}_j \hat{k}_4 \ldots \hat{k}_4\}^{(n)} e_{ij}(m), \tag{A-B-32}$$

where e_{ij} is the traceless symmetric $O(3)$ spin two polarization tensor. One then deduces that

$$\begin{aligned} &\{\hat{k}_i \hat{k}_j \hat{k}_4 \ldots \hat{k}_4\}^{(n)} - \tfrac{1}{3}\delta_{ij} \{\hat{k}_\ell \hat{k}_\ell \hat{k}_4 \ldots \hat{k}_4\}^{(n)} \\ &= \frac{1}{2^{n-3}\, n(n-1)} C^3_{n-2}(\cos\beta) \left[\hat{k}_i \hat{k}_j - \tfrac{1}{3}\delta_{ij} \hat{k}_\ell \hat{k}_\ell\right], \end{aligned} \tag{A-B-33}$$

where the repeated index ℓ is summed from 1 to 3. Noting that

$$\hat{k}_\ell \hat{k}_\ell = \sin^2\beta$$

and that

$$\{\hat{k}_\ell \hat{k}_\ell \hat{k}_4 \ldots \hat{k}_4\}^{(n)} = -\{\hat{k}_4 \ldots \hat{k}_4\}^{(n)},$$

the above expression can be simplified. Making use of the recursion relations for the Gegenbauer polynomials, it can be reduced to

$$\begin{aligned} \{\hat{k}_i \hat{k}_j \hat{k}_4 \ldots \hat{k}_4\}^{(n)} &= \frac{1}{n(n-1)2^{n-3}} \\ &\times \left[C^3_{n-2}(\cos\beta)\hat{k}_i \hat{k}_j - \tfrac{1}{4} C^2_{n-2}(\cos\beta)\delta_{ij}\right]. \end{aligned} \tag{A-B-34}$$

In this form, the relation can be proven rather easily by induction.

It will be useful to express $\{\hat{k}_\mu \hat{k}_4 \ldots \hat{k}_4\}^{(n)}$ and $\{\hat{k}_\mu \hat{k}_\nu \hat{k}_4 \ldots \hat{k}_4\}^{(n)}$ in terms of Gegenbauer polynomials. All the necessary information is contained in eq. (A-B-29), (-31), and (-34). To express the results compactly, define the four vector

$$\eta \equiv (0, 0, 0, 1). \tag{A-B-35}$$

Then, with the help of the Gegenbauer polynomial recursion relations, it can be shown that

$$\{\hat{k}_\mu \hat{k}_4 \ldots \hat{k}_4\}^{(n)} = \frac{1}{2^{n-1} n} \Big[C^2_{n-1}(\cos\beta)\, \hat{k}_\mu - C^2_{n-2}(\cos\beta)\, \eta_\mu \Big], \tag{A-B-36}$$

and

$$\{\hat{k}_\mu \hat{k}_\nu \hat{k}_4 \ldots \hat{k}_4\}^{(n)} = \frac{1}{2^{n-3} n(n-1)} \Big[C^3_{n-2}(\cos\beta)\, \hat{k}_\mu \hat{k}_\nu - C^3_{n-3}(\cos\beta)(\eta_\mu \hat{k}_\nu + \hat{k}_\mu \eta_\nu) + C^3_{n-4}(\cos\beta)\, \eta_\mu \eta_\nu - \tfrac{1}{4} C^2_{n-2}(\cos\beta)\, \delta_{\mu\nu} \Big] \tag{A-B-37}$$

The two previous equations, as well as the recursion relations (A-B-4) - (A-B-8), are to be interpreted using the convention that

$$C^\alpha_n(x) = 0 \quad \text{for} \quad n < 0. \tag{A-B-38}$$

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BIOGRAPHICAL NOTE

Alan Guth was born February 27, 1947, in New Brunswick, New Jersey. He grew up in Highland Park, New Jersey. He entered M.I.T. in September, 1964, and has been a student there until the completion of this thesis. In June, 1969, he received the degrees of S.B. and S.M. in physics. His masters thesis, "The Design of High Momentum Transfer Electrodisintegration Experiments", was done under the supervision of Prof. Aron Bernstein. Mr. Guth was a Full-Time Teaching Assistant from September, 1969, to January, 1971. He held the Karl Taylor Compton Fellowship during most of the rest of his graduate student career, and is also an Honorary Woodrow Wilson Fellow. Since September, 1971, he has been an Instructor in the Department of Physics, Princeton University.

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